Curvature and Entropy Perturbations in Generalized Gravity

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Abstract

We investigate the cosmological perturbations in generalized gravity, where the Ricci scalar and a scalar field are non-minimally coupled via an arbitrary function $f(\varphi, R)$. In the Friedmann-Lemaître-Robertson-Walker (FLRW) background, by studying the linear perturbation theory, we separate the scalar type perturbations into the curvature perturbation and the entropy perturbation, whose evolution equations are derived. Then we apply this framework to inflation. We consider the generalized slow-roll conditions and the quantization initial condition. Under these conditions, two special examples are studied analytically. One example is the case with no entropy perturbation. The other example is a model with the entropy perturbation large initially but decaying significantly after crossing the horizon.

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I. INTRODUCTION

In the past three decades, huge progress has been made on our understanding of the early universe, both theoretically and observationally. This is implemented by the inflation theory [1, 2, 3] merging the general relativity and quantum field theory in an elegant way. On the one hand, inflation theory has naturally explained the initial condition of big bang cosmology. On the other hand, it also makes quantitative predictions which can be tested by precise observational data [4, 5, 6].

So far the prevail inflation models are based on the Einstein gravity coupled minimally to a scalar field (or more scalar fields) [7, 8, 9, 10, 11, 12, 13, 14]. Whereas considerable investigations were also performed on models of modified gravity with higher derivative corrections or non-minimal coupling. Most of them can be classified into two categories:

- $f(R)$ models without a scalar field [15, 16];
- $F(\varphi)R$ scalar-tensor theory.

Each of them has only one degree of freedom. This is clear if one takes a conformal transformation as done by [17, 18, 19].

Our intention in this paper is to deal with a general type of model, namely $f(\varphi, R)$ gravity, which unifies and generalizes the above relatively simpler models. The action of this model is of the form[1]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\varphi, R) - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V(\varphi)\right]. \tag{1}$$

Here the $f(\varphi, R)$ term contains a non-minimal coupling between the scalar field φ and the Ricci scalar R. While $V(\varphi)$ is the potential of the scalar field. In principle $V(\varphi)$ can be absorbed in $f(\varphi, R)$, but we will keep them separate.

In contrast with simpler models, the $f(\varphi, R)$ model usually introduces another degree of freedom. Generally speaking, due to the new degree of freedom, there is a non-vanishing entropy perturbation in most models based on $f(\varphi, R)$ gravity. In the present work, we will distinguish the entropy perturbation from curvature perturbation, and then study their evolutions.

To make our discussion self-contained and clear in notations, first of all, we collect some previously known results in section II. Our general result is presented in section III, where we extract the curvature perturbation and entropy perturbation as well as their evolution equations. Applying this formalism to inflation, we study the generalized slow-roll conditions and the quantized initial condition in section IV. In section V, some typical examples are studied under the slow-roll approximation. One example is the case with no entropy perturbation, including the simpler models with one degree of freedom we mentioned above. The other example is to add a $g(\varphi)R^2$ correction to Einstein gravity. Specifically, we study the $g(\varphi) = \frac{1}{4}\lambda\varphi^2$, $V(\varphi) = \frac{1}{2}m^2\varphi^2$ model in the limit $M_p^2/\varphi^2 \ll \lambda m^2\varphi^2/M_p^2 \ll 1$. Ignoring the coupling of perturbations inside the Hubble horizon, we find the entropy perturbation is large at horizon-crossing but decays significantly outside the horizon. Initially the correlation between the curvature perturbation and the entropy perturbation has been neglected under

[1] Throughout this paper, we employ the reduced Planck mass $M_p = 1/\sqrt{8\pi G}$ and set $\hbar = c = 1$.

our approximation, but at the end of inflation they become moderately anti-correlated. We summarize and refer to some open problems in section VI. For reference and as a support to our canonical quantized initial conditions, in appendix A we collect the relevant results of a two-field model which is conformally equivalent to the $f(\varphi, R)$ model. In the generalized gravity, since it is difficult to draw a clear borderline between gravity and the matter, there is ambiguity in defining curvature perturbation and entropy perturbation. we present a more traditional (but less tractable) definition of these perturbations in appendix B. Complementary to section III, details for deriving the evolution equation of entropy perturbation are relegated to appendix C.

II. REVIEW OF PREVIOUSLY KNOWN RESULTS

The cosmological perturbations in $f(\varphi, R)$ gravity has been studied actively by Hwang and Noh in [20, 21, 22, 23, 24] *etc.* But most of the investigations mainly concentrated on theories with one degree of freedom, including the $f(R)$ theory and the scalar-tensor theory. For generalized $f(\varphi, R)$ theory, the complete evolution equations of the first order perturbations were obtained in [24], where the background dynamics were also shown. In this section, to make our discussion self-contained and clear in notations, we write down these results following the notations of [7], except for that the metric signature is taken to be $(-,+,+,+)$. All of the results collected here can be found in [24, 25]. Throughout this paper, we will focus on the scalar type perturbations, working in the longitudinal gauge. The tensor type perturbation has been addressed in [23]. There was a discussion of scalar type perturbations in [22], though, for generalized gravity, their full evolution equations were obtained in [24]. In this section, we summarize these results in a self-contained way, and at the same time, set up our convention of notations. In the subsequent sections, we will take further steps to get some new results.

The background is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 = a^2(\tau)(-d\tau^2 + d\vec{x}^2), \tag{2}$$

where t is the comoving time and τ is the conformal time, with respect to which the derivatives will be denoted by a dot overhead and a superscript prime respectively. Later on, for the sake of convenience, we will also use a superscript "$\bullet$" to denote the derivative with respect to t. Then in terms of the Hubble parameter $H = \dot{a}/a$, the Ricci scalar[2] can be expressed as $R = 6(2H^2 + \dot{H})$. Throughout this paper, we will only deal with the flat universe. The results for closed or open universe are expected to be similar.

For succinctness let us introduce the notation $F = \frac{\partial}{\partial R} f(\varphi, R)$. In this paper we will concentrate on the case with $F > 0$, but it is easy to extend our results to the case $F \leq 0$. The fluctuation of the scalar field φ will be denoted by $\delta\varphi$. It is well known that the perturbations of the metric can be decomposed into three types: the scalar type, the vector type and the tensor type. In the scalar-driven inflation, these three types are decoupled with each other if we only consider two-point correlation functions. Here we are interested in the linear order scalar type perturbations, so we can treat them exclusively, not worrying

[2] When defining the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, we take the convention of Ricci tensor $R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\kappa}\Gamma^\kappa_{\mu\nu} - \Gamma^\lambda_{\nu\kappa}\Gamma^\kappa_{\mu\lambda}$ with the affine connections $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$.

about the vector type and tensor type perturbations. Considering scalar type perturbations only, the perturbed metric takes the form

$$ds^2 = -(1+2\phi)dt^2 + 2a\partial_i\mathcal{B}dtdx^i + a^2[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j\mathcal{E}]dx^idx^j. \tag{3}$$

Here δ_{ij} is the Kronecker delta function. We will mainly work in the longitudinal gauge, that is, choosing $\mathcal{B} = \mathcal{E} = 0$. If necessary, one can easily recover all of our results into the gauge-invariant form with the following dictionary [7, 24]:

$$\begin{aligned}
\phi &\rightarrow \phi^{(gi)} = \phi + \frac{[a(\mathcal{B}-\mathcal{E}')]'}{a},\\
\psi &\rightarrow \psi^{(gi)} = \psi - aH(\mathcal{B}-\mathcal{E}'),\\
\delta\varphi &\rightarrow \delta\varphi^{(gi)} = \delta\varphi + a\dot{\varphi}(\mathcal{B}-\mathcal{E}'),\\
\delta F &\rightarrow \delta F^{(gi)} = \delta F + a\dot{F}(\mathcal{B}-\mathcal{E}').
\end{aligned} \tag{4}$$

Corresponding to action (1), the variation of φ and $g_{\mu\nu}$ gives

$$\begin{aligned}
\delta_1 S \;=\; & \int d^4x\mathfrak{D}_0 + \int d^4x\left[\sqrt{-g}\left(\frac{1}{2}f_{,\varphi} - V_{,\varphi}\right) + \partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi)\right]\delta\varphi\\
& + \int d^4x\Bigg\{\frac{1}{2}\sqrt{-g}\Bigg[-g_{\mu\nu}\left(\frac{1}{2}f - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V\right) + FR_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi\\
& +\partial_\alpha F g^{\alpha\beta}\frac{1}{2}(\partial_\beta g_{\mu\nu} + \partial_\mu g_{\nu\beta} - \partial_\nu g_{\mu\beta}) - \partial_\mu F g^{\alpha\beta}\frac{1}{2}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})\Bigg]\\
& +\partial_\kappa\left[\frac{1}{4}\sqrt{-g}\partial_\lambda F(2g^{\alpha\beta}g^{\lambda\kappa} - g^{\alpha\lambda}g^{\beta\kappa} - g^{\alpha\kappa}g^{\beta\lambda})\right]g_{\alpha\mu}g_{\beta\nu}\Bigg\}\delta g^{\mu\nu},
\end{aligned} \tag{5}$$

where the total derivative term

$$\begin{aligned}
\mathfrak{D}_0 \;=\; & \partial_\lambda\left(\frac{1}{2}\sqrt{-g}Fg^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right) - \partial_\nu\left(\frac{1}{2}\sqrt{-g}Fg^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right) - \partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\delta\varphi)\\
& +\partial_\kappa\left[\frac{1}{4}\sqrt{-g}\partial_\lambda F(2g^{\mu\nu}g^{\lambda\kappa} - g^{\mu\lambda}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\lambda})\delta g_{\mu\nu}\right].
\end{aligned} \tag{6}$$

Then the generalized Friedmann equations are simply

$$\frac{1}{2}\dot{\varphi}^2 + V - \frac{1}{2}f + 3H^2F + 3\dot{H}F - 3H\dot{F} = 0, \tag{7}$$

$$\dot{\varphi}^2 + 2\dot{H}F - H\dot{F} + \ddot{F} = 0. \tag{8}$$

Equation (8) proves to be very useful and we will use it frequently without mention. The non-minimal coupling $f(\varphi, R)$ brings a new term to the equation of motion for the scalar field

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{2}f_{,\varphi} + V_{,\varphi} = 0. \tag{9}$$

From (5), we get the Einstein equations of linear perturbations[24, 25] in the longitudinal

gauge

$$3H(\dot{\psi}+H\phi)-\frac{1}{a^2}\nabla^2\psi=-\frac{1}{2M_p^2}\delta\rho, \tag{10}$$

$$\dot{\psi}+H\phi=-\frac{1}{2M_p^2}\delta q, \tag{11}$$

$$\psi-\phi=\frac{\delta F}{F}, \tag{12}$$

$$\ddot{\psi}+3H(H\phi+\dot{\psi})+H\dot{\phi}+2\dot{H}\phi+\frac{1}{3a^2}\nabla^2(\phi-\psi)=\frac{1}{2M_p^2}\delta p, \tag{13}$$

in which $\delta\rho$, δq and δp are defined as

$$\begin{aligned}
\delta\rho &= \frac{M_p^2}{F}[\dot{\varphi}\delta\dot{\varphi}+\frac{1}{2}(-f_{,\varphi}+2V_{,\varphi})\delta\varphi-3H\delta\dot{F}+(3\dot{H}+3H^2+\frac{1}{a^2}\nabla^2)\delta F \\
&\quad +(3H\dot{F}-\dot{\varphi}^2)\phi+3\dot{F}(H\phi+\dot{\psi})], \\
\partial_i(\delta q) &= -\frac{M_p^2}{F}\partial_i(\dot{\varphi}\delta\varphi+\delta\dot{F}-H\delta F-\dot{F}\phi), \\
\delta p &= \frac{M_p^2}{F}[\dot{\varphi}\delta\dot{\varphi}+\frac{1}{2}(f_{,\varphi}-2V_{,\varphi})\delta\varphi+\delta\ddot{F}+2H\delta\dot{F}+(-\dot{H}-3H^2-\frac{2}{3a^2}\nabla^2)\delta F \\
&\quad -\dot{F}\dot{\phi}-(\dot{\varphi}^2+2\ddot{F}+2H\dot{F})\phi-2\dot{F}(H\phi+\dot{\psi})].
\end{aligned} \tag{14}$$

Equations (10-13) follow respectively from the G_0^0, G_i^0, $G_j^i(i\neq j)$, G_i^i components of Einstein equations. The equation of motion for $\delta\varphi$ gives a redundant relation.

Using equations (11) and (12) to cancel $\delta\varphi$ and δF respectively in (10), one obtains

$$\begin{aligned}
&F(\ddot{\phi}+\ddot{\psi})+(HF+3\dot{F}-\frac{2F\ddot{\varphi}}{\dot{\varphi}})(\dot{\phi}+\dot{\psi}) \\
&+[(3H\dot{F}+3\ddot{F})-\frac{2\ddot{\varphi}}{\dot{\varphi}}(HF+2\dot{F})-\frac{F}{a^2}\nabla^2]\phi \\
&+[(4\dot{H}F+H\dot{F}-\ddot{F})-\frac{2\ddot{\varphi}}{\dot{\varphi}}(HF-\dot{F})-\frac{F}{a^2}\nabla^2]\psi=0.
\end{aligned} \tag{15}$$

Inserting (11) and (12) into the G_i^i component equation (13), it will result in a relation automatically satisfied by the background equation (9).

Remembering that $\delta F=F_{,R}\delta R+F_{,\varphi}\delta\varphi$ and $\dot{F}=F_{,R}\dot{R}+F_{,\varphi}\dot{\varphi}$, we can write (12) in another form

$$\begin{aligned}
F(\phi-\psi) &= 2F_{,R}[3\ddot{\psi}+3H\dot{\phi}+12H\dot{\psi}+6(2H^2+\dot{H})\phi+\frac{1}{a^2}\nabla^2(\phi-2\psi)] \\
&\quad -\frac{F_{,\varphi}}{\dot{\varphi}}[F(\dot{\phi}+\dot{\psi})+(HF+2\dot{F})\phi+(HF-\dot{F})\psi].
\end{aligned} \tag{16}$$

In minimally coupled model $f=M_p^2R$, equation (16) reduces to the familiar relation $\phi=\psi$.

Comments are needed here. We start with five equations (four perturbed Einstein equations and one equation of motion for $\delta\varphi$) of three variables ($\delta\varphi$, ϕ, ψ), so this system of equations appears to be over-determined. However, as we have just mentioned, the equation

of motion for $\delta\varphi$ is redundant, which can be derived from perturbed Einstein equations. In addition, one of the remaining four equations turns out to be automatically satisfied by background equations. This is also understandable if one recalls the Bianchi identity. Consequently, there are three independent equations with three variable at last. What we have done was just using one of the equations to cancel $\delta\varphi$, and arriving at two equations (15), (16) with two variables ϕ, ψ.

III. CURVATURE PERTURBATION AND ENTROPY PERTURBATION

The perturbation equations (15) and (16) were used in [24, 25] to study certain $f(\varphi, R)$ models with only one degree of freedom, where the effects of entropy perturbation have been neglected all the while. However, in the most general case, the entropy perturbation may play an important role in the early stage and then decay or translate into the curvature perturbation, leaving some observable signatures in fluctuations of cosmic microwave background (CMB) and dark matter.[3] Disregarding the entropy perturbation would obscure many interesting phenomenological predictions, such as the residual entropy perturbation and the large non-Gaussianity. From this point of view, for generalized $f(\varphi, R)$ gravity, it is important to take the new degree of freedom into consideration and study the entropy perturbation seriously.

In this section, we will rearrange the perturbation equations (15) and (16) in order to decompose the scalar type perturbations into curvature and entropy components and to get their evolution equations. In subsequent sections, applied to inflation, the evolution dynamics of perturbations will be studied under the slow-roll approximation.

For our following study, it is essential to notice that equation (15) can be recast in the form

$$\dot{\mathcal{R}} = \left(\ln \frac{\dot{\varphi}^2}{2F\dot{\varphi}^2 + 3\dot{F}^2}\right)^{\bullet} \frac{2HF + \dot{F}}{2F\dot{\varphi}^2 + 3\dot{F}^2}\delta s + \frac{2HF + \dot{F}}{2F\dot{\varphi}^2 + 3\dot{F}^2}\frac{F}{a^2}\nabla^2(\phi + \psi), \tag{17}$$

with the curvature perturbation

$$\mathcal{R} = \frac{1}{2}(\phi + \psi) + \frac{2HF + \dot{F}}{2F\dot{\varphi}^2 + 3\dot{F}^2}\left[F(\dot{\phi} + \dot{\psi}) + \frac{1}{2}(2HF + \dot{F})(\phi + \psi)\right]. \tag{18}$$

The first term on the right hand side of (17) can be taken as the entropy perturbation, [4] or more exactly, the relative entropy perturbation [13]

$$\mathcal{S} = \frac{\dot{F}(2HF + \dot{F})}{\dot{\varphi}(2F\dot{\varphi}^2 + 3\dot{F}^2)}\sqrt{\frac{3}{2F}}\delta s, \tag{19}$$

[3] A mechanism for entropy perturbation decay into curvature perturbation was realized in the curvaton scenario [26, 27]. In $f(\varphi, R)$ gravity one may expect a similar story. It is remarkable that by curvaton mechanism a large non-Gaussianty can be generated [26, 27, 28, 29, 30].

[4] Here is an ambiguity in normalizing entropy perturbation. We define the entropy perturbation by (19), but we will abuse it for δs and $\delta\tilde{s}$ since they are proportional to $\mathcal{S}$ up to background quantities. Through relations (19) and (28) they are easy to be traded with each other. Our normalization of (19) is chosen to ensure that $\mathcal{P}_{\mathcal{R}*} = \mathcal{P}_{\mathcal{S}*}$ when perturbations cross the Hubble horizon, as given by equation (72).

where

$$\delta s \; = \; F(\dot{\phi}+\dot{\psi})+\frac{1}{2}(2HF+\dot{F})(\phi+\psi)+\frac{2F\dot{\varphi}^2+3\dot{F}^2}{2\dot{F}}(\phi-\psi). \tag{20}$$

Strictly speaking, the second term on the right hand side of (17) also contributes to the total entropy perturbation, but it is suppressed on super-horizon scale [13]. Throughout our paper, we focus on the relative entropy perturbation, and refer it as entropy perturbation for simplicity. The adiabatic (curvature) and isocurvature (entropy) perturbations were investigated in [31, 32, 33, 34] for a two-field Lagrangian with a non-standard kinetic term. The Lagrangian discussed there is equivalent to the $f(\varphi, R)$ theory here, up to a conformal transformation [18, 21]. One can check that the curvature perturbation defined in (18) is conformally equivalent to that appeared in [31, 32, 33, 34]. See also appendix A. As another check, in the minimally coupled limit $f = M_p^2 R$, the expression (18) reduces to the familiar form $\mathcal{R} = \phi - H(\dot{\phi} + H\phi)/\dot{H}$.

Along the line of [35], we give an apparently more traditional definition of curvature perturbation $\mathcal{R}_{eff}$ and entropy perturbation δs_{eff} in appendix B. At the first glance, the traditional definition seems more physical. But it depends heavily on an artificial separation of gravity and matter content. The choice given by (18) and (20) is more convenient in calculation. Moreover, the curvature and entropy perturbations at the end of inflation are not necessarily the ones probed by astronomical observations [5, 6], because they may evolve significantly after the exit of inflation, depending on the details of reheating. We will take a pragmatic attitude and prefer the convenient definition (18) and (20) here.

One may still raise a question: why do we claim that $\mathcal{R}$ and $\mathcal{S}$ defined above correspond to curvature perturbation and entropy perturbation respectively? This doubt can be resolved by rewriting (18) and (20) into the following form:

$$\begin{aligned} \mathcal{R} \; &= \; \psi + \frac{2HF+\dot{F}}{2F\dot{\varphi}^2+3\dot{F}^2}\dot{\varphi}\delta\varphi + \frac{3H\dot{F}-\dot{\varphi}^2}{2F\dot{\varphi}^2+3\dot{F}^2}\delta F, \\ \delta s \; &= \; \dot{\varphi}^2\left(\frac{\delta\varphi}{\dot{\varphi}} - \frac{\delta F}{\dot{F}}\right). \end{aligned} \tag{21}$$

When deriving these relations, we have used equations (8), (11) and (12).

First of all, with (21) at hand, we can apply it to the special limits mentioned in section I:

- $f(R)$ models without a scalar field;
- $F(\varphi)R$ scalar-tensor theory.

These models are extensively studied in the past [20, 21, 22, 23, 24]. For $f(R)$ models, $\dot{\varphi}^2 = 0$, thus the entropy perturbation vanishes obviously. As for scalar-tensor theory, since F becomes a function exclusively dependent of φ, we have

$$\frac{\delta F}{\dot{F}} = \frac{\delta\varphi}{\dot{\varphi}}, \tag{22}$$

which guarantees $\delta s = 0$. Now it becomes very clear that there is no entropy perturbation in these models. This is in agreement with the fact that there is only one degree of freedom

in these models. On the other hand, curvature perturbation $\mathcal{R}$ reduces to

$$\mathcal{R} = \begin{cases} \psi + \frac{H}{\dot{\varphi}}\delta\varphi, & \text{for pure } f(R) \text{ gravity without scalar field;} \\ \psi + \frac{H}{\dot{F}}\delta F, & \text{for } f = F(\varphi)R. \end{cases} \tag{23}$$

These are exactly the ones having appeared in [24]. Our definition of curvature perturbation $\mathcal{R}$ naturally generalizes them in a unified form.

Secondly, in favor of the small dictionary (4), one can prove that the definitions of $\mathcal{R}$ and δs are gauge-invariant. In other words, the $(\mathcal{B} - \mathcal{E}')$ terms cancel out automatically.

As a further check, since our definition of curvature perturbation is gauge-invariant, by choosing a special (but not longitudinal) gauge

$$(2HF + \dot{F})\dot{\varphi}\delta\varphi^{(c)} + (3H\dot{F} - \dot{\varphi}^2)\delta F^{(c)} = 0, \tag{24}$$

we get a neat relation

$$\mathcal{R} = \psi^{(c)}. \tag{25}$$

In minimally coupled models, the same relation (25) holds in comoving gauge or uniform density gauge $\delta\varphi^{(c)} = 0$. Here the gauge condition (24) generalizes the comoving gauge condition, so we can take it as a generalized comoving gauge. In fact, just as in minimal models, relation (25) has a geometric interpretation. The spatial curvature to the first order of perturbations is given by

$${}^{(3)}R = \frac{4}{a^2}\psi^{(c)}. \tag{26}$$

Therefore, in gauge (24), the adiabatic perturbation $\mathcal{R}$ is proportional to the spatial curvature ${}^{(3)}R$. This is why we give it the name "curvature perturbation".

According to our definition in this section, the entropy perturbation is proportional to $\frac{\delta\varphi}{\dot{\varphi}} - \frac{\delta F}{\dot{F}}$. This is natural if one remembers that a new scalar degree of freedom is related to F in the present theory. It gets dynamical when F becomes dynamical. This form is very similar to the entropy perturbation in models with two fields, say φ_1, φ_2, corresponding to an entropy perturbation proportional to $\frac{\delta\varphi_1}{\dot{\varphi}_1} - \frac{\delta\varphi_2}{\dot{\varphi}_2}$.

We would like to emphasize that the perturbations defined in appendix B make senses only if one views the $f(\varphi, R)$ theory effectively as Einstein gravity with exotic matter contents. This viewpoint is not so reliable since the gravity has been modified in $f(\varphi, R)$ model. Furthermore, what we are really interested in is the evolution of ϕ and ψ. As we will show below, the choice in this section is powerful to study their evolution.

In appendix C, we demonstrate that the evolution of entropy perturbation obeys the

equation

$$\begin{aligned}
\ddot{\delta s} \;=\; & \left\{\left[\ln\frac{\dot\varphi^2(2F\dot\varphi^2+3\dot F^2)}{\dot F^2}\right]^\bullet - 3H\right\}\dot{\delta s} + \frac{\nabla^2}{a^2}\delta s \\
& +\left\{\left(\frac{F\dot\varphi^2}{\dot F}\right)^{\bullet\bullet} + \left(\frac{3\dot F\ddot\varphi}{\dot\varphi} - \dot\varphi^2 - \frac{3}{2}\ddot F\right)^\bullet + \left(\frac{2\ddot\varphi}{\dot\varphi} - \frac{3\dot F}{2F}\right)\left(\frac{3\dot F\ddot\varphi}{\dot\varphi} - \dot\varphi^2 - 3\ddot F\right)\right. \\
& + \left(\frac{\dot\varphi^2}{\dot F} + \frac{3\dot F}{2F}\right)\left[2F(2H^2+\dot H) + \frac{3\dot F\ddot\varphi}{\dot\varphi} - \dot\varphi^2 - 3\ddot F - \frac{F^2}{3F_{,R}} - \frac{F\dot F F_{,\varphi}}{2F_{,R}\dot\varphi}\right] \\
& \left. + \left[\left(\ln\frac{\dot\varphi^2(2F\dot\varphi^2+3\dot F^2)}{\dot F^2}\right)^\bullet - 3H\right]\left[\dot\varphi^2 + \frac{3}{2}\ddot F - \left(\frac{F\dot\varphi^2}{\dot F}\right)^\bullet - \frac{3\dot F\ddot\varphi}{\dot\varphi}\right]\right\} \\
& \times\frac{2\dot F}{2F\dot\varphi^2+3\dot F^2}\delta s \\
& + \left[\frac{2F\dot\varphi^2}{3\dot F} + F\left(\ln\frac{\dot F^2}{2F\dot\varphi^2+3\dot F^2}\right)^\bullet\right]\frac{\nabla^2}{a^2}(\phi+\psi).
\end{aligned} \tag{27}$$

In terms of

$$\delta\tilde s = \frac{\dot F}{\dot\varphi\sqrt{4F\dot\varphi^2+6\dot F^2}}\delta s, \tag{28}$$

this equation can be reexpressed as

$$\begin{aligned}
& \ddot{\delta\tilde s} + 3H\dot{\delta\tilde s} - \frac{\nabla^2}{a^2}\delta\tilde s \\
= \; & \frac{1}{2}\left[\ln\frac{\dot\varphi^2(2F\dot\varphi^2+3\dot F^2)}{\dot F^2}\right]^\bullet\left\{\left[\frac{3}{2}\ln\frac{\dot\varphi^2(2F\dot\varphi^2+3\dot F^2)}{\dot F^2}\right]^\bullet\right. \\
& \left. -3H - \left[\ln\left(\frac{\dot\varphi^2(2F\dot\varphi^2+3\dot F^2)}{\dot F^2}\right)^\bullet\right]^\bullet\right\}\delta\tilde s \\
& +\left\{\left(\frac{F\dot\varphi^2}{\dot F}\right)^{\bullet\bullet} + \left(\frac{3\dot F\ddot\varphi}{\dot\varphi} - \dot\varphi^2 - \frac{3}{2}\ddot F\right)^\bullet + \left(\frac{2\ddot\varphi}{\dot\varphi} - \frac{3\dot F}{2F}\right)\left(\frac{3\dot F\ddot\varphi}{\dot\varphi} - \dot\varphi^2 - 3\ddot F\right)\right. \\
& + \left(\frac{\dot\varphi^2}{\dot F} + \frac{3\dot F}{2F}\right)\left[2F(2H^2+\dot H) + \frac{3\dot F\ddot\varphi}{\dot\varphi} - \dot\varphi^2 - 3\ddot F - \frac{F^2}{3F_{,R}} - \frac{F\dot F F_{,\varphi}}{2F_{,R}\dot\varphi}\right] \\
& \left. + \left[\left(\ln\frac{\dot\varphi^2(2F\dot\varphi^2+3\dot F^2)}{\dot F^2}\right)^\bullet - 3H\right]\left[\dot\varphi^2 + \frac{3}{2}\ddot F - \left(\frac{F\dot\varphi^2}{\dot F}\right)^\bullet - \frac{3\dot F\ddot\varphi}{\dot\varphi}\right]\right\} \\
& \times\frac{2\dot F}{2F\dot\varphi^2+3\dot F^2}\delta\tilde s \\
& + \left[\frac{2F\dot\varphi^2}{3\dot F} + F\left(\ln\frac{\dot F^2}{2F\dot\varphi^2+3\dot F^2}\right)^\bullet\right]\frac{\dot F}{\dot\varphi\sqrt{4F\dot\varphi^2+6\dot F^2}}\frac{\nabla^2}{a^2}(\phi+\psi).
\end{aligned} \tag{29}$$

At the same time, according to (17), the evolution equation of $\mathcal{R}$ is

$$\dot{\mathcal{R}}=\left(\ln\frac{\dot{\varphi}^2}{2F\dot{\varphi}^2+3\dot{F}^2}\right)^{\bullet}\frac{2\dot{\varphi}(2HF+\dot{F})}{\dot{F}\sqrt{4F\dot{\varphi}^2+6\dot{F}^2}}\delta\tilde{s}+\frac{2HF+\dot{F}}{2F\dot{\varphi}^2+3\dot{F}^2}\frac{F}{a^2}\nabla^2(\phi+\psi), \tag{30}$$

There are similar equations in non-standard two-field models [31, 32, 33, 34]. In principle, equations (29) and (30) may be obtained by a conformal transformation from the counterparts in [31, 32, 33, 34]. However, having avoided the intricacies of conformal transformation at the perturbation level, our derivation in the $f(\varphi, R)$ frame is straightforward. Nonetheless, it is still interesting to compare the results here and those in [31, 32, 33, 34] via the conformal transformation [18, 21]. Such a comparison would confirm the conformal equivalence at the perturbation level.

Until now we have not made any approximation. Equations (29) and (30) determine the dynamics of entropy and curvature perturbations exactly. They are applicable in various cosmological stages and scenarios of FLRW universe. Given a concrete model, the classical evolution of perturbations can be numerically followed utilizing these equations. To work in the inflation scenario and generate an appropriate spectrum of density fluctuation, we should consider the generalized slow-roll conditions and the initial condition. This is a task of the next section.

IV. SLOW-ROLL APPROXIMATION AND QUANTIZATION

In this section we will study the perturbations under the generalized slow-roll approximation. Firstly we study the classical evolution. At the end of this section, we will discuss the quantization of perturbations as an initial condition.

From equation (18), we know that the curvature perturbation $\mathcal{R}$ is fully determined by $\phi+\psi$ and its time derivative. So the dynamics of $\phi+\psi$ informs us the dynamics of $\mathcal{R}$. Inserting (18) into (17), one obtains

$$\begin{aligned}\frac{1}{F}\left(\ln\frac{2F\dot{\varphi}^2+3\dot{F}^2}{\dot{\varphi}^2}\right)^{\bullet}\delta s+(\ddot{\phi}+\ddot{\psi})+\left(\ln\frac{aF^3}{2F\dot{\varphi}^2+3\dot{F}^2}\right)^{\bullet}(\dot{\phi}+\dot{\psi})\\+\frac{1}{2}[\ln(a^2F)]^{\bullet}\left[\ln\frac{(2HF+\dot{F})^2}{2F\dot{\varphi}^2+3\dot{F}^2}\right]^{\bullet}(\phi+\psi)-\frac{\nabla^2}{a^2}(\phi+\psi)\;=\;0.\end{aligned} \tag{31}$$

We observe that substituting this equation into (27) will result in a fourth order differential equation of $\phi+\psi$. But that equation would be rather difficult to solve directly. In this section, we rewrite equations (27) and (31) into a tractable form. In the next section, we will get their analytical solutions in some special examples.

Firstly, let us define two new variables:

$$u_{\mathcal{R}}=\frac{F^{\frac{3}{2}}}{\sqrt{4F\dot{\varphi}^2+6\dot{F}^2}}(\phi+\psi),\qquad u_{\mathcal{S}}=a\delta\tilde{s}. \tag{32}$$

Similar to $\phi+\psi$, the variable $u_{\mathcal{R}}$ tells us the evolution of curvature perturbation $\mathcal{R}$. Following

relation (31), it obeys the equation of motion

$$\frac{a\dot{\varphi}F^{\frac{1}{2}}}{\dot{F}}\left(\ln\frac{2F\dot{\varphi}^2+3\dot{F}^2}{\dot{\varphi}^2}\right)^{\bullet}u_{\mathcal{S}}+u''_{\mathcal{R}}-\nabla^2u_{\mathcal{R}}+m^2_{\mathcal{R}}a^2u_{\mathcal{R}}=0, \tag{33}$$

with the "mass squared"

$$\begin{aligned} m^2_{\mathcal{R}} \;=\; & \frac{F^{\frac{3}{2}}}{\sqrt{2F\dot{\varphi}^2+3\dot{F}^2}}\left(\frac{\sqrt{2F\dot{\varphi}^2+3\dot{F}^2}}{F^{\frac{3}{2}}}\right)^{\bullet\bullet}-\frac{1}{2}\left(\ln\frac{2F\dot{\varphi}^2+3\dot{F}^2}{aF^3}\right)^{\bullet}\left(\ln\frac{2F\dot{\varphi}^2+3\dot{F}^2}{F^3}\right)^{\bullet} \\ & +\frac{1}{2}[\ln(a^2F)]^{\bullet}\left[\ln\frac{(2HF+\dot{F})^2}{2F\dot{\varphi}^2+3\dot{F}^2}\right]^{\bullet}. \end{aligned} \tag{34}$$

If we introduce the notation

$$E=\frac{2HF+\dot{F}}{F^{\frac{3}{2}}}, \tag{35}$$

then one can easily prove that

$$\dot{E}=-\frac{2F\dot{\varphi}^2+3\dot{F}^2}{2F^{\frac{5}{2}}}. \tag{36}$$

During the inflation stage, the Hubble parameter and the effective energy density are almost constants. In the following, we will study perturbation dynamics under the slow-roll approximation. For the convenience of calculation, our definition of slow-roll parameters is

$$\begin{aligned} &\epsilon_1=\frac{\dot{H}}{H^2}, \quad \epsilon_2=\frac{\ddot{H}}{H\dot{H}}, \quad \eta_1=\frac{\ddot{\varphi}}{H\dot{\varphi}}, \\ &\eta_2=\frac{\dddot{\varphi}}{H\ddot{\varphi}}, \quad \delta_1=\frac{\dot{F}}{HF}, \quad \delta_2=\frac{\dot{E}}{HE}, \\ &\delta_3=\frac{\ddot{F}}{H\dot{F}}, \quad \delta_4=\frac{\ddot{E}}{H\dot{E}}, \quad \delta_6=\frac{\dddot{E}}{H\ddot{E}}\ . \end{aligned} \tag{37}$$

While the slow-roll conditions are given by

$$|\epsilon_i|\ll 1, \quad |\eta_i|\ll 1, \quad |\delta_i|\ll 1. \tag{38}$$

Be careful with the notation and sign difference between the slow-roll parameters here and those in most literature. Under the slow-roll conditions, the background equations (7-9) are significantly simplified,

$$V-\frac{1}{2}f+3H^2F\simeq 0, \tag{39}$$

$$\dot{\varphi}^2+2\dot{H}F-H\dot{F}\simeq 0, \tag{40}$$

$$3H\dot{\varphi}-\frac{1}{2}f_{,\varphi}+V_{,\varphi}\simeq 0. \tag{41}$$

Notice that equation (41) can be consistently derived from the leading order Friedmann equations (39) and (40). Neglecting higher order terms, we have

$$\begin{aligned} &\dot{E}=-\frac{\dot{\varphi}^2}{F^{\frac{3}{2}}}, \quad \ddot{E}=\frac{3\dot{F}\dot{\varphi}^2-4F\dot{\varphi}\ddot{\varphi}}{2F^{\frac{5}{2}}}, \quad \delta_2\simeq\epsilon_1-\frac{1}{2}\delta_1, \\ &\delta_4\simeq 2\eta_1-\frac{3}{2}\delta_1, \quad \delta_6\simeq\eta_1-\frac{5}{2}\delta_1+\frac{3\delta_1\delta_3-\delta_1\eta_1-4\eta_1\eta_2}{3\delta_1-4\eta_1}. \end{aligned} \tag{42}$$

Under the slow-roll approximation, the Hubble parameter is almost constant during inflation. In terms of slow-roll parameters, the nondimensionalized "mass squared" for $u_{\mathcal{R}}$ is

$$\frac{m_{\mathcal{R}}^2}{H^2} \simeq 2\epsilon_1 - \eta_1. \tag{43}$$

Therefore, to the leading order, the evolution equation of $u_{\mathcal{R}}$ is

$$u''_{\mathcal{R}k} + k^2 u_{\mathcal{R}k} + m_{\mathcal{R}}^2 a^2 u_{\mathcal{R}k} + \beta u_{\mathcal{S}k} = 0 \tag{44}$$

in the Fourier space. Here we have used the notation

$$\beta \simeq \mathrm{sign}(\dot\varphi) aH \sqrt{\delta_1 - 2\epsilon_1} \tag{45}$$

with $\mathrm{sign}(\dot\varphi) = \dot\varphi/|\dot\varphi|$. In other words, the sign of β depends on the sign of $\dot\varphi$.

At the same time, in terms of $u_{\mathcal{S}}$, the evolution equation (29) of entropy perturbation can be written to the leading order as

$$u''_{\mathcal{S}k} + k^2 u_{\mathcal{S}k} + m_{\mathcal{S}}^2 a^2 u_{\mathcal{S}k} + \alpha k^2 u_{\mathcal{R}k} = 0, \tag{46}$$

in which

$$\frac{m_{\mathcal{S}}^2}{H^2} \simeq \frac{5}{2}\delta_1 - 5\epsilon_1 + \frac{F}{3H^2 F_{,R}} + \frac{\dot F F_{,\varphi}}{2H^2 F_{,R}\dot\varphi} - 6, \tag{47}$$

$$\frac{\alpha}{aH} = \frac{\dot F}{H\dot\varphi F^{\frac{1}{2}}}\left[\frac{2\dot\varphi^2}{3\dot F} + \left(\ln\frac{\dot F^2}{2F\dot\varphi^2 + 3\dot F^2}\right)^{\bullet}\right] \simeq \mathrm{sign}(\dot\varphi)\frac{2}{3}\sqrt{\delta_1 - 2\epsilon_1}. \tag{48}$$

In the above, we discussed the classical perturbations and a generalized slow-roll condition. The initial condition is governed by quantum theory of fluctuations. That is, we should expand action (1) to the second order with respect to perturbations, taking the form

$$\delta_2 S = \int d^3\vec x d\tau \left[\frac{1}{2}(\partial_\tau v_{\mathcal{R}})^2 + \frac{1}{2}(\partial_\tau v_{\mathcal{S}})^2 - \frac{1}{2}(\partial_i v_{\mathcal{R}})^2 - \frac{1}{2}(\partial_i v_{\mathcal{S}})^2 + C_J v_{\mathcal{R}}\partial_\tau v_{\mathcal{S}} - V(v_{\mathcal{R}}, v_{\mathcal{S}})\right]. \tag{49}$$

In this action the covariant variables $v_{\mathcal{R}}$ and $v_{\mathcal{S}}$ are linear superpositions of perturbations. Then the canonical quantization leads to the initial condition

$$v_{\mathcal{R}k} \to \frac{1}{\sqrt{2k}} e^{-ik\tau}\hat e_{\mathcal{R}k}, \qquad v_{\mathcal{S}k} \to \frac{1}{\sqrt{2k}} e^{-ik\tau}\hat e_{\mathcal{S}k} \tag{50}$$

in the short wavelength limit $k/(aH) \to \infty$. Here $\{\hat e_{\mathcal{R}k}, \hat e_{\mathcal{S}k}\}$ is the orthogonal basis

$$\langle \hat e_{\alpha k}, \hat e_{\beta k'}\rangle = \delta_{\alpha\beta}\delta(k - k'), \qquad \alpha, \beta = \mathcal{R}, \mathcal{S}. \tag{51}$$

From the argument in appendix A, we believe the canonical variables in $f(\varphi, R)$ gravity should be

$$\begin{aligned} v_{\mathcal{R}} &= a\sqrt{\frac{2F}{2F\dot\varphi^2 + 3\dot F^2}}\left[\dot\varphi\delta\varphi + 3\dot F\psi + \frac{F(\dot\varphi^2 - 3H\dot F)}{2HF + \dot F}(\phi + \psi)\right] \\ &= a\sqrt{\frac{2F}{2F\dot\varphi^2 + 3\dot F^2}}\frac{2F\dot\varphi^2 + 3\dot F^2}{2HF + \dot F}\mathcal{R}, \\ v_{\mathcal{S}} &= \frac{\sqrt{3}a}{\sqrt{2F\dot\varphi^2 + 3\dot F^2}}[\dot F\delta\varphi + F\dot\varphi(\phi - \psi)] = \frac{\sqrt{3}a\dot F}{\dot\varphi\sqrt{2F\dot\varphi^2 + 3\dot F^2}}\delta s = \sqrt{6}u_{\mathcal{S}}. \end{aligned} \tag{52}$$

A detailed form of (49) and a closer analysis of canonical quantization will be presented elsewhere.

When solving equations (44) and (46), the initial condition (50) for $v_{\mathcal{R}}$ and $v_{\mathcal{S}}$ sets the initial condition of $u_{\mathcal{R}}$ and $u_{\mathcal{S}}$. In the short wavelength limit, we find $v_{\mathcal{R}k} \to 2u'_{\mathcal{R}k}$. Thus we have

$$u_{\mathcal{R}k} \to \frac{1}{(2k)^{\frac{3}{2}}} e^{i\left(\frac{\pi}{2}-k\tau\right)} \hat{e}_{\mathcal{R}k}, \qquad u_{\mathcal{S}k} \to \frac{1}{\sqrt{12k}} e^{-ik\tau} \hat{e}_{\mathcal{S}k}. \tag{53}$$

One can check that this solution meets (44) and (46) well in the limit $k/(aH) \to \infty$.

V. ANALYTICAL EXAMPLES

In equations (44) and (46), it is clear that $u_{\mathcal{R}k}$ and $u_{\mathcal{S}k}$ are coupled, while the coupling strength is controlled by α and β. In general the coupled equations are difficult to solve analytically, thus numerical approaches are needed. Nevertheless, under some circumstances, analytical results are available. Let us focus on the case $\delta s = 0$ in subsection V A. In subsection V B, we discuss the $g(\varphi)R^2$ model with a non-vanishing entropy perturbation. Under certain approximation, we solve a special case analytically, resulting in nearly scale-invariant power spectra.

A. No Relative Entropy Perturbation: $\delta s = 0$

This is essentially the case with only one degree of freedom. Therefore, when $\delta s = 0$, equation (46) is expected to be satisfied automatically, while the dynamics of the survival degree of freedom is described by equation (44). Although we cannot give a general proof to this claim, we can check it in two classes of models mentioned previously:

- $f(R)$ models without a scalar field;
- $F(\varphi)R$ scalar-tensor theory.

For $f(R)$ models, this is quite clear because $u_{\mathcal{S}} = 0$ and $\alpha = 0$. For $F(\varphi)R$ scalar-tensor theory, it is less obvious. But looking at equation (47), one may notice that we have assumed $F_{,R} \neq 0$ when writing down (46) and there is a singularity in $m^2_{\mathcal{S}}$. Multiplying equation (46) with $F_{,R}$ to eliminate the illegal singularity in $m^2_{\mathcal{S}}$, we will find the resulting equation is well-defined and automatically satisfied.

We can not exclude the possibility that the above claim does not hold in some cases, although so far we can not come up with such an example. In those cases, equation (46) gives $u_{\mathcal{R}} = 0$, which also satisfies (44). But then there will be no quantum fluctuations. So we are not interested in this possibility even if it exists.

Hence let us assume equation (46) is satisfied more generally in the cases we are interested with $\delta s = 0$. Then we are left with one differential equation of $u_{\mathcal{R}}$,

$$u''_{\mathcal{R}k} + k^2 u_{\mathcal{R}k} + m^2_{\mathcal{R}} a^2 u_{\mathcal{R}k} = 0, \tag{54}$$

which is obtained by setting $u_{\mathcal{S}} = 0$ in equation (44).

In accordance with initial condition (53), we find the solution for (54) is

$$u_{\mathcal{R}k} = -\frac{1}{4k^{\frac{3}{2}}} e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} \sqrt{-\pi k\tau} H^{(1)}_{\nu}(-k\tau) = C\sqrt{z} H^{(1)}_{\nu}(z), \qquad \text{with } \nu^2 = \frac{1}{4} - \frac{m^2_{\mathcal{R}}}{H^2}. \tag{55}$$

Here we introduced the notation $z = -k\tau$. This form of solution is consistent with the minimal inflation model [8]. For $\nu \simeq \frac{1}{2}$, it gives a nearly scale-invariant power spectrum. The coefficient C is determined by quantization condition (53). Keep in mind that in the limit $z \to \infty$,

$$H_\nu^{(1)}(z) \to \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \propto e^{-ik\tau}. \tag{56}$$

According to inflation theory, the CMB fluctuations are seeded by primordial quantum fluctuations inside the horizon in the early universe. When the primordial fluctuations crossed the Hubble horizon, they began the classical evolution phase. So the initial condition for classical evolution is dictated by quantization. Here we did not study the quantization condition carefully. It would be important to put this result on firmer ground by quantizing the second order action directly.

If the scale factor is exponentially growing $a \sim e^{Ht}$, then there is a relation

$$aH = -\frac{1}{\tau} = \frac{k}{z}. \tag{57}$$

Straightforward calculations give

$$\begin{aligned} u'_{\mathcal{R}k} &= -Ck\left[\sqrt{z}H_{\nu-1}^{(1)}(z) + \frac{1-2\nu}{2\sqrt{z}}H_\nu^{(1)}(z)\right], \\ u''_{\mathcal{R}k} &= -Ck^2\left(\sqrt{z} + \frac{1-4\nu^2}{4z^{\frac{3}{2}}}\right)H_\nu^{(1)}(z). \end{aligned} \tag{58}$$

Notice that a prime denotes the derivative with respect to conformal time τ.

In the long wavelength limit $k\tau \to 0$, for $\nu \simeq \frac{1}{2}$, we obtain to the leading order of slow-roll parameters,

$$\begin{aligned} \mathcal{R}_k &\simeq \frac{1}{4k^{\frac{3}{2}}}\left(\frac{2HF + \dot{F}}{F}\right)^2 \sqrt{\frac{F}{2F\dot{\varphi}^2 + 3\dot{F}^2}}\left(\frac{-k\tau}{2}\right)^{\frac{1}{2}-\nu} \\ &\simeq \frac{H^2}{k^{\frac{3}{2}}}\sqrt{\frac{F}{2F\dot{\varphi}^2 + 3\dot{F}^2}}. \end{aligned} \tag{59}$$

As a result, the power spectrum of curvature perturbation is

$$\mathcal{P}_\mathcal{R} = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 = \frac{H^4 F}{2\pi^2(2F\dot{\varphi}^2 + 3\dot{F}^2)}, \tag{60}$$

and thus the spectral index is

$$n_\mathcal{R} - 1 = \frac{4\dot{H}}{H^2} + \frac{\dot{F}}{HF} - \frac{(2F\dot{\varphi}^2 + 3\dot{F}^2)^\bullet}{H(2F\dot{\varphi}^2 + 3\dot{F}^2)}\ . \tag{61}$$

Recalling that in [24] Hwang and Noh gave the corresponding result

$$n_\mathcal{R} - 1 \simeq \begin{cases} \frac{4\dot{H}}{H^2} + \frac{\dot{F}}{HF} - \frac{2\ddot{F}}{H\dot{F}}, & \text{for pure } f(R) \text{ gravity without scalar field;} \\ \frac{4\dot{H}}{H^2} + \frac{\dot{F}}{HF} - \frac{(2F\dot{\varphi}^2+3\dot{F}^2)^\bullet}{H(2F\dot{\varphi}^2+3\dot{F}^2)}, & \text{for } f = F(\varphi)R. \end{cases} \tag{62}$$

Obviously the final results (61) and (62) are perfectly matched. Notice that for the case of pure $f(R)$ gravity, we can recover (62) from (61) by setting $\dot{\varphi} = 0$.

In the leading order, we also find $n_{\mathcal{R}} - 1 = 1 - 2\nu$. This indicates $\dot{\mathcal{R}}_k/(H\mathcal{R}_k) \simeq 0$ for long wavelength perturbations. That is to say, under the generalized slow-roll approximation, the curvature perturbation is almost conserved outside the horizon. On the one hand, this extends the previous result in [24] to the general case $\delta s = 0$ (and equation (46) satisfied automatically). On the other hand, to get richer phenomena, we should take the entropy perturbation into account and solve evolution equations (44) and (46) more generally.

B. $g(\varphi)R^2$ Correction

A relatively simple but non-trivial model with non-vanishing entropy perturbation is to consider the R^2 correction with a φ-dependent coefficient,

$$f(\varphi, R) = M_p^2 R + g(\varphi) R^2. \tag{63}$$

For this class of model, under the slow-roll approximation, equations (39-41) become

$$V \simeq 3M_p^2 H^2, \quad \dot{\varphi}^2 + 2\dot{H}F - H\dot{F} \simeq 0, \quad 3H\dot{\varphi} \simeq 72 g_{,\varphi} H^4 - V_{,\varphi}. \tag{64}$$

Then we get the following relations:

$$\begin{aligned}
&F = M_p^2 + 2gR \simeq M_p^2 + \frac{8gV}{M_p^2}, \\
&V_{,\varphi}\dot{\varphi} \simeq 6M_p^2 H\dot{H} = 2HV\epsilon_1, \\
&V_{,\varphi}\ddot{\varphi} + V_{,\varphi\varphi}\dot{\varphi}^2 \simeq 6M_p^2(H\ddot{H} + \dot{H}^2), \\
&3H\ddot{\varphi} + 3\dot{H}\dot{\varphi} \simeq \left(\frac{8g_{,\varphi}V^2}{M_p^4} - V_{,\varphi}\right)_{,\varphi} \dot{\varphi}, \\
&3H\dddot{\varphi} + 6\dot{H}\ddot{\varphi} + 3\ddot{H}\dot{\varphi} \simeq \left(\frac{8g_{,\varphi}V^2}{M_p^4} - V_{,\varphi}\right)_{,\varphi} \ddot{\varphi} + \left(\frac{8g_{,\varphi}V^2}{M_p^4} - V_{,\varphi}\right)_{,\varphi\varphi} \dot{\varphi}^2.
\end{aligned} \tag{65}$$

The slow-roll parameters (37) can be expressed in terms of g and V and their derivatives

with respect to φ,

$$
\begin{aligned}
\epsilon_1 &= \frac{\dot{H}}{H^2} \simeq \frac{4g_{,\varphi}V_{,\varphi}}{M_p^2} - \frac{M_p^2 V_{,\varphi}^2}{2V^2}, \\
\eta_1 &= \frac{\ddot{\varphi}}{H\dot{\varphi}} \simeq \frac{8g_{,\varphi\varphi}V}{M_p^2} + \frac{12g_{,\varphi}V_{,\varphi}}{M_p^2} - \frac{M_p^2 V_{,\varphi\varphi}}{V} + \frac{M_p^2 V_{,\varphi}^2}{2V^2}, \\
\delta_1 &= \frac{\dot{F}}{HF} \simeq \frac{16\epsilon_1 V(gV)_{,\varphi}}{V_{,\varphi}(M_p^4 + 8gV)}, \qquad \delta_2 = \frac{\dot{E}}{HE} \simeq \epsilon_1 - \frac{1}{2}\delta_1, \\
\delta_3 &= \frac{\ddot{F}}{H\dot{F}} = \eta_1 + \frac{2\epsilon_1 V(gV)_{,\varphi\varphi}}{V_{,\varphi}(gV)_{,\varphi}}, \qquad \delta_4 = \frac{\ddot{E}}{H\dot{E}} \simeq 2\eta_1 - \frac{3}{2}\delta_1, \\
\epsilon_2 &= \frac{\ddot{H}}{H\dot{H}} \simeq \eta_1 - \epsilon_1 + \frac{2\epsilon_1 V V_{,\varphi\varphi}}{V_{,\varphi}^2}, \\
\eta_2 &= \frac{\dddot{\varphi}}{H\ddot{\varphi}} \simeq \eta_1 - \epsilon_1 - \frac{\epsilon_1 \ddot{H}}{\eta_1 H\dot{H}} + \frac{2\epsilon_1 M_p^2}{\eta_1 V_{,\varphi}} \left(\frac{8g_{,\varphi}V^2}{M_p^4} - V_{,\varphi}\right)_{,\varphi\varphi}, \\
\delta_6 &= \frac{\dddot{E}}{H\ddot{E}} \simeq \eta_1 - \frac{5}{2}\delta_1 + \frac{3\delta_1\delta_3 - \delta_1\eta_1 - 4\eta_1\eta_2}{3\delta_1 - 4\eta_1}.
\end{aligned} \tag{66}
$$

Subsequently, the coefficients in equations (44) and (46) are

$$
\begin{aligned}
m_{\mathcal{R}}^2 &\simeq H^2(2\epsilon_1 - \eta_1), \qquad \beta \simeq \mathrm{sign}(\dot{\varphi}) aH\sqrt{\delta_1 - 2\epsilon_1}, \qquad \alpha \simeq \mathrm{sign}(\dot{\varphi})\frac{2}{3}aH\sqrt{\delta_1 - 2\epsilon_1}, \\
m_{\mathcal{S}}^2 &\simeq H^2\left[\frac{5}{2}\delta_1 - 3\epsilon_1 + \frac{M_p^4}{2gV} - 2 + \frac{48g_{,\varphi}(gV)_{,\varphi}}{gM_p^2}\right].
\end{aligned} \tag{67}
$$

The above results are still too complicated to get some sense. Particularly, since there are so many slow-roll parameters, one may even worry about whether the slow-roll conditions (38) can be satisfied simultaneously. However, for the special case[5]

$$
g(\varphi) = \frac{1}{4}\lambda\varphi^2, \qquad V(\varphi) = \frac{1}{2}m^2\varphi^2, \tag{68}
$$

the slow-roll parameters take much simpler form as below:

$$
\begin{aligned}
\epsilon_1 &= \frac{\dot{H}}{H^2} \simeq \frac{2\lambda m^2\varphi^2}{M_p^2} - \frac{2M_p^2}{\varphi^2}, \qquad \eta_1 = \frac{\ddot{\varphi}}{H\dot{\varphi}} \simeq \frac{8\lambda m^2\varphi^2}{M_p^2}, \\
\delta_1 &= \frac{\dot{F}}{HF} \simeq \frac{4\epsilon_1\lambda m^2\varphi^4}{M_p^4 + \lambda m^2\varphi^4}, \qquad \delta_2 = \frac{\dot{E}}{HE} \simeq \epsilon_1 - \frac{1}{2}\delta_1, \\
\delta_3 &= \frac{\ddot{F}}{H\dot{F}} = \eta_1 + 3\epsilon_1, \qquad \delta_4 = \frac{\ddot{E}}{H\dot{E}} \simeq 2\eta_1 - \frac{3}{2}\delta_1, \\
\epsilon_2 &= \frac{\ddot{H}}{H\dot{H}} \simeq \eta_1, \qquad \eta_2 = \frac{\dddot{\varphi}}{H\ddot{\varphi}} \simeq \eta_1 + 3\epsilon_1, \\
\delta_6 &= \frac{\dddot{E}}{H\ddot{E}} \simeq 3\epsilon_1 + 2\eta_1 - \frac{5}{2}\delta_1 + \frac{\delta_1\eta_1}{4\eta_1 - 3\delta_1}.
\end{aligned} \tag{69}
$$

[5] As a matter of fact, a model with $R^2\varphi^2$ correction was discussed in [36].

We have been concentrating on the case $F > 0$, so in this example we limit our attention to the case with $\lambda > 0$. Obviously the slow-roll conditions (38) are satisfied when $M_p^2 \ll \varphi^2 \ll M_p^2/(\lambda m^2)$. To meet this condition we should fine-tune the coupling constants to be very small $\lambda m^2 \ll 1$. This is the large field inflation. Although the value of field φ is larger than Planck mass, thanks to the small coupling constants, its energy density is still less than the Planck energy density.

In the above special case, if we further assume $M_p^2/\varphi^2 \ll \lambda m^2 \varphi^2/M_p^2 \ll 1$ during inflation, then we will find

$$\begin{aligned}
&\epsilon_1 \simeq \frac{2\lambda m^2 \varphi^2}{M_p^2}, \qquad \eta_1 \simeq \delta_1 \simeq 4\epsilon_1, \\
&m_{\mathcal{R}}^2 \simeq -2\epsilon_1 H^2, \qquad \beta \simeq \mathrm{sign}(\dot{\varphi}) a H \sqrt{2\epsilon_1}, \\
&m_{\mathcal{S}}^2 \simeq (31\epsilon_1 - 2) H^2, \qquad \alpha \simeq \mathrm{sign}(\dot{\varphi}) \frac{2}{3} a H \sqrt{2\epsilon_1}.
\end{aligned} \tag{70}$$

The model with coefficients (70) is relatively simple. Let us discuss it in some detail.

Again, due to the non-vanishing α and β, the interaction terms form an obstacle to our analytical study. However, it is still interesting to make some rough estimates by neglecting these interactions before horizon-crossing/Hubble-exit. The problem is akin to the one we met in inflation of coupled multiple field. In fact, there is an excellent analysis of coupling effects in [37]. It turned out if the coupling terms are of slow-roll order, then they will give a correction of slow-roll order compared to the leading order contribution. In our case, the coefficients of coupling terms are of order $\mathcal{O}(\sqrt{\epsilon_1})$, so we expect their corrections to the power spectra are suppressed by $\mathcal{O}(\sqrt{\epsilon_1})$.

Our scheme is taking limit $M_p^2/\varphi^2 \ll \lambda m^2 \varphi^2/M_p^2 \ll 1$, and disregarding the interaction terms related to α and β inside the horizon $k/(aH) \gtrsim 1$. In accordance with initial condition (53), we get an analytical solution to equations (44) and (46),

$$\begin{aligned}
u_{\mathcal{R}k} &= -\frac{1}{4k^{\frac{3}{2}}} e^{i(\nu_1 - \frac{1}{2})\frac{\pi}{2}} \sqrt{-\pi k\tau} H_{\nu_1}^{(1)}(-k\tau) \hat{e}_{\mathcal{R}k}, \qquad \text{with } \nu_1^2 = \frac{1}{4} - \frac{m_{\mathcal{R}}^2}{H^2}, \\
u_{\mathcal{S}k} &= -\frac{1}{2\sqrt{6k}} e^{i(\nu_2 - \frac{3}{2})\frac{\pi}{2}} \sqrt{-\pi k\tau} H_{\nu_2}^{(1)}(-k\tau) \hat{e}_{\mathcal{S}k}, \qquad \text{with } \nu_2^2 = \frac{1}{4} - \frac{m_{\mathcal{S}}^2}{H^2}.
\end{aligned} \tag{71}$$

From this solution, we find at the time of horizon-crossing, the power spectra of curvature and entropy perturbations are nearly scale-invariant,

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}*} &= \frac{k^3}{2\pi^2} |\mathcal{R}_{k*}|^2 \simeq \left. \frac{H^4}{4\pi^2 \dot{\varphi}^2} \right|_* \simeq \frac{M_p^2}{96\pi^2 \lambda^2 m^2 \varphi_*^4}, \\
\mathcal{P}_{\mathcal{S}*} &= \frac{k^3}{2\pi^2} |\mathcal{S}_{k*}|^2 \simeq \left. \frac{H^4}{4\pi^2 \dot{\varphi}^2} \right|_* \simeq \frac{M_p^2}{96\pi^2 \lambda^2 m^2 \varphi_*^4},
\end{aligned} \tag{72}$$

while their cross-correlation is negligible,[6]

$$\mathcal{P}_{\mathcal{C}*} = \frac{k^3}{2\pi^2} \langle \mathcal{R}_{k*}, \mathcal{S}_{k*} \rangle \simeq 0. \tag{73}$$

[6] This is because we have neglected the coupling terms in (44) and (46) before Hubble-crossing. The fact is, when taking coupling terms into consideration, we expect the cross-correlation is not negligible here, $\mathcal{P}_{\mathcal{C}*}/\mathcal{P}_{\mathcal{S}*} \sim \mathcal{O}(\sqrt{\epsilon_1})$. But to get the explicit value of it, one should either perform the higher order calculation or take a numerical method.

Therefore the spectral indices at the horizon-crossing are

$$n_{\mathcal{R}*} - 1 = n_{\mathcal{S}*} - 1 = 4\epsilon_{1*} - 2\eta_{1*} = -4\epsilon_{1*}. \tag{74}$$

The variables with an asterisk subscript take their values at the horizon-crossing time $k = aH$. Note we have chosen the normalization of $\mathcal{S}$ in (19) so that $\mathcal{P}_{\mathcal{R}*} = \mathcal{P}_{\mathcal{S}*}$.

We would like to pause here and comment on the scale invariance of the entropy perturbation. If readers go through our calculation carefully, they would find the limit $M_p^2/\varphi^2 \ll \lambda m^2\varphi^2/M_p^2 \ll 1$ is of key importance in making $\mathcal{P}_{\mathcal{S}*}$ scale-invariant. In this limit, the $M_p^4/(2gV)$ term is small, hence $m_{\mathcal{S}}^2/H^2 \simeq -2$ and subsequently $\nu_2 \simeq 3/2$. If $M_p^4/(2gV)$ is not small enough, we cannot get such a value of ν_2 and then the entropy power spectrum will not be scale-invariant. Even in the limit we have taken, there are subtleties with the $M_p^4/(2gV)$ term. If $M_p^4/(2gV) \sim \mathcal{O}(\epsilon_1)$, we should pick it up when calculating $m_{\mathcal{S}}^2$. However, for simplicity of our following estimation, when writing down (70), we have assumed $M_p^4/(2gV) \ll \epsilon_1$.

If one considers interactions outside the horizon, it proves useful to describe the evolution of perturbations in terms of a general transfer matrix

$$\begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{R}\mathcal{S}} \\ 0 & T_{\mathcal{S}\mathcal{S}} \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix}_*, \tag{75}$$

which gives the power spectra of scalar type perturbations at the end of inflation

$$\mathcal{P}_{\mathcal{R}} = \mathcal{P}_{\mathcal{R}*} + T_{\mathcal{R}\mathcal{S}}^2 \mathcal{P}_{\mathcal{S}*}, \qquad \mathcal{P}_{\mathcal{S}} = T_{\mathcal{S}\mathcal{S}}^2 \mathcal{P}_{\mathcal{S}*}, \qquad \mathcal{P}_{\mathcal{C}} = T_{\mathcal{R}\mathcal{S}} T_{\mathcal{S}\mathcal{S}} \mathcal{P}_{\mathcal{S}*}. \tag{76}$$

These are the power spectra probed by astronomical observation [5, 6], unless they changed significantly from the end of inflation to the matter-radiation decoupling.

Although the cross-correlation power spectrum is negligibly small when the perturbations cross the Hubble horizon, it may still be large at the end of inflation. This is is usually evaluated by the cross-correlation coefficient $\tilde{\beta}$ [12] (here we use a tilde to distinguish it from the notation appeared in equation (44)) or the correlation angle Δ [14]:

$$\tilde{\beta} = \cos\Delta = \frac{\mathcal{P}_{\mathcal{C}}}{\sqrt{\mathcal{P}_{\mathcal{R}}\mathcal{P}_{\mathcal{S}}}}. \tag{77}$$

In terms of the entropy-to-curvature ratio

$$r_{\mathcal{S}} = \frac{\mathcal{P}_{\mathcal{S}}}{\mathcal{P}_{\mathcal{R}}}, \tag{78}$$

it is simplified by the transfer relation (76),

$$\cos\Delta = \mathrm{sign}(T_{\mathcal{S}\mathcal{S}}) T_{\mathcal{R}\mathcal{S}} \sqrt{\frac{r_{\mathcal{S}*}}{1 + T_{\mathcal{R}\mathcal{S}}^2 r_{\mathcal{S}*}}} = \frac{\mathrm{sign}(T_{\mathcal{S}\mathcal{S}}) T_{\mathcal{R}\mathcal{S}}}{\sqrt{1 + T_{\mathcal{R}\mathcal{S}}^2}}. \tag{79}$$

Here we have used the notation $\mathrm{sign}(T_{\mathcal{S}\mathcal{S}}) = T_{\mathcal{S}\mathcal{S}}/|T_{\mathcal{S}\mathcal{S}}|$ and the fact $r_{\mathcal{S}*} \simeq 1$. Following the line of [37], we expect the coupling terms inside the horizon would introduce $\mathcal{O}(\sqrt{\epsilon_1})$ corrections to $r_{\mathcal{S}*}$.

As a matter of fact, well after the Hubble-exit $k/(aH) \ll 1$, the interaction term $\alpha k^2 u_{\mathcal{R}k}$ in equation (46) is negligible, thus the entropy perturbation evolves independently outside the

horizon. As usually done in literature, when dealing with the evolution equation of entropy perturbation, one can ignore the second order term (*i.e.* $\ddot{\mathcal{S}}_k$ term) and terms proportional to k^2. Recalling relations (19), (52) and

$$u_{\mathcal{S}} = \frac{a\sqrt{F(2F\dot{\varphi}^2 + 3\dot{F}^2)}}{\sqrt{3}(2HF + \dot{F})}\mathcal{S}, \tag{80}$$

from equation (46) we can get the evolution equation of $\mathcal{S}$ to the leading order,

$$\dot{\mathcal{S}}_k = -\frac{H}{3}\left(\frac{m_{\mathcal{S}}^2}{H^2} + 2 + \epsilon_1 - \frac{9}{4}\delta_1 - 3\delta_2 + \frac{3}{2}\delta_4\right)\mathcal{S}_k = \mu_{\mathcal{S}} H \mathcal{S}_k. \tag{81}$$

which is a first order differential equation because the second order term has been ignored. This equation, together with (17) outside the horizon

$$\begin{aligned}
\dot{\mathcal{R}}_k &= \frac{\dot{\varphi}}{\dot{F}}\sqrt{\frac{2F}{3}}\left(\ln\frac{\dot{\varphi}^2}{2F\dot{\varphi}^2 + 3\dot{F}^2}\right)^{\bullet}\mathcal{S}_k \\
&= \frac{\mathrm{sign}(\dot{\varphi})}{\delta_1}\sqrt{\frac{2(\delta_1 - 2\epsilon_1)}{3}}\left(2\eta_1 - \frac{5}{2}\delta_1 - \delta_4\right)H\mathcal{S}_k \\
&= \mu_{\mathcal{R}} H \mathcal{S}_k,
\end{aligned} \tag{82}$$

dictates the transfer matrix (75). When evaluating $m_{\mathcal{S}}^2$ in equation (81), we should keep the slow-roll order quantities in (47) because $\mu_{\mathcal{S}}$ is of the slow-roll order. Taking $\mu_{\mathcal{S}}$ and $\mu_{\mathcal{R}}$ as their average values between the horizon-exit and the end of inflation, we can quickly write down the solution for (81) and (82):

$$\begin{aligned}
\mathcal{S}_k &= \mathcal{S}_{k*}\exp\left(\int_{t_*}^{t}\mu_{\mathcal{S}} H dt\right) = \mathcal{S}_{k*}e^{\mu_{\mathcal{S}}(N_* - N)}, \\
\mathcal{R}_k - \mathcal{R}_{k*} &= \int_{t_*}^{t}\mu_{\mathcal{R}} H \mathcal{S}_k dt = \int \frac{\mu_{\mathcal{R}}}{\mu_{\mathcal{S}}}\mathcal{S}_{k*} d e^{-\int \mu_{\mathcal{S}} dN} = \frac{\mu_{\mathcal{R}}}{\mu_{\mathcal{S}}}\mathcal{S}_{k*}\left[e^{\mu_{\mathcal{S}}(N_* - N)} - 1\right],
\end{aligned} \tag{83}$$

in which $N = \ln[a_{end}/a(t)]$ stands for the e-folding number from time t to the end of inflation. Using this solution one may check that $\ddot{\mathcal{S}}_k/(H\dot{\mathcal{S}}_k) \sim \mathcal{O}(\epsilon_1)$. So it was reasonable for us to ignore the $\ddot{\mathcal{S}}_k$ term.

With the above results and formulas, it is not hard to obtain the power spectra at the end of inflation:

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}} &\simeq \mathcal{P}_{\mathcal{R}*} + \mathcal{P}_{\mathcal{S}*}\frac{\mu_{\mathcal{R}}^2}{\mu_{\mathcal{S}}^2}\left[e^{\mu_{\mathcal{S}}(N_* - N)} - 1\right]^2, \\
\mathcal{P}_{\mathcal{S}} &\simeq \mathcal{P}_{\mathcal{S}*}e^{2\mu_{\mathcal{S}}(N_* - N)}, \\
\mathcal{P}_{\mathcal{C}} &\simeq \mathcal{P}_{\mathcal{S}*}\frac{\mu_{\mathcal{R}}}{\mu_{\mathcal{S}}}e^{\mu_{\mathcal{S}}(N_* - N)}\left[e^{\mu_{\mathcal{S}}(N_* - N)} - 1\right].
\end{aligned} \tag{84}$$

The spectral indices at the end of inflation are

$$\begin{aligned}
n_{\mathcal{R}} - 1 &= n_{\mathcal{S}*} - 1 - \frac{2\mu_{\mathcal{R}}^2\mu_{\mathcal{S}}e^{\mu_{\mathcal{S}}(N_* - N)}\left[e^{\mu_{\mathcal{S}}(N_* - N)} - 1\right]}{\mu_{\mathcal{S}}^2 + \mu_{\mathcal{R}}^2\left[e^{\mu_{\mathcal{S}}(N_* - N)} - 1\right]^2}, \\
n_{\mathcal{S}} - 1 &= n_{\mathcal{S}*} - 1 - 2\mu_{\mathcal{S}}, \\
n_{\mathcal{C}} - 1 &= n_{\mathcal{S}*} - 1 - \frac{\mu_{\mathcal{S}}\left[2e^{\mu_{\mathcal{S}}(N_* - N)} - 1\right]}{e^{\mu_{\mathcal{S}}(N_* - N)} - 1}.
\end{aligned} \tag{85}$$

In model (68) with $M_p^2/\varphi^2 \ll \lambda m^2\varphi^2/M_p^2 \ll 1$, we find $3H\dot{\varphi} > 0$, so $\text{sign}(\dot{\varphi}) = 1$. Under the slow-roll approximation, we get

$$\mu_{\mathcal{R}} \simeq -\sqrt{\frac{4\epsilon_1}{3}}, \qquad \mu_{\mathcal{S}} \simeq -\frac{29}{3}\epsilon_1, \qquad r_{\mathcal{S}*} \simeq 1. \tag{86}$$

If we estimate $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{R}}$ with their values at Hubble-exit, the final result will be very simple (taking $N = 0$ at the end of inflation):

$$\begin{aligned}
&\frac{\mathcal{P}_{\mathcal{R}}}{\mathcal{P}_{\mathcal{S}*}} \simeq 1 + \frac{12}{841\epsilon_{1*}}\left(e^{-29\epsilon_{1*}N_*/3} - 1\right)^2, \qquad \frac{\mathcal{P}_{\mathcal{S}}}{\mathcal{P}_{\mathcal{S}*}} \simeq e^{-58\epsilon_{1*}N_*/3}, \\
&\frac{\mathcal{P}_{\mathcal{C}}}{\mathcal{P}_{\mathcal{S}*}} \simeq \frac{2}{29}\sqrt{\frac{3}{\epsilon_{1*}}}e^{-29\epsilon_{1*}N_*/3}\left(e^{-29\epsilon_{1*}N_*/3} - 1\right), \\
&n_{\mathcal{R}} - 1 = -4\epsilon_{1*}, \qquad n_{\mathcal{S}} - 1 = \frac{46}{3}\epsilon_{1*}, \qquad n_{\mathcal{C}} - 1 = \frac{17}{3}\epsilon_{1*}.
\end{aligned} \tag{87}$$

At the end of inflation, the curvature perturbation and the entropy perturbation are moderately anti-correlated, with the cross-correlation coefficient

$$\tilde{\beta} = \cos\Delta \simeq -\sqrt{\frac{12}{841\epsilon_{1*} + 12}}, \tag{88}$$

but the entropy-to-curvature ratio is small enough,

$$r_{\mathcal{S}} \simeq \frac{841\epsilon_{1*}e^{-58\epsilon_{1*}N_*/3}}{841\epsilon_{1*} + 12}. \tag{89}$$

The evolution of power spectra of curvature and entropy perturbations and their correlation have been shown in figure 1. We can see the power spectrum of curvature perturbation, denoted by the blue solid line, almost doubled from the horizon-crossing (at $N_* - N \simeq 0$) to the end of inflation (at $N_* - N \simeq 60$). Although the entropy perturbation (denoted by the purple dashed line) was lager than the curvature perturbation at the Hubble-crossing, it was decaying exponentially with respect to $N_* - N$ in the super-horizon scale. The correlation between curvature perturbation and entropy perturbation is depicted by the brown dot-dashed line. Once crossing out the horizon, it kept a negative value, with its amplitude at first increasing and then decreasing.

In figure 2, we illustrate the dependence of correlation coefficient (the top graph), the logarithm of entropy-to-curvature ratio (the middle graph) and tensor-to-scalar ratio (the bottom graph) on $N_* - N$. It is clear that the entropy-to-curvature ratio dropped down quickly outside the horizon. The entropy perturbation and the curvature perturbation was totally uncorrelated (under the decoupled approximation) at the Hubble-exit but evolved to be moderately anti-correlated at the end of inflation.

At the end of inflation, if one assumes $n_{\mathcal{R}} - 1 \simeq -0.04$, then $r_{\mathcal{S}} \simeq 4 \times 10^{-6}$. This is well below the upper bound put by the five-year Wilkinson Microwave Anisotropy Probe (WMAP) [5]. Of course, we cannot take these numbers seriously, because our above analytical results are meaningful only for estimation. The coupling between entropy perturbation and curvature perturbation has been systematically ignored until horizon-crossing. In addition we have estimated $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{R}}$ with their Hubble-exit values. Remember that in writing down (70) we have neglected the $M_p^4/(2gV)$ term. This may also have introduced some

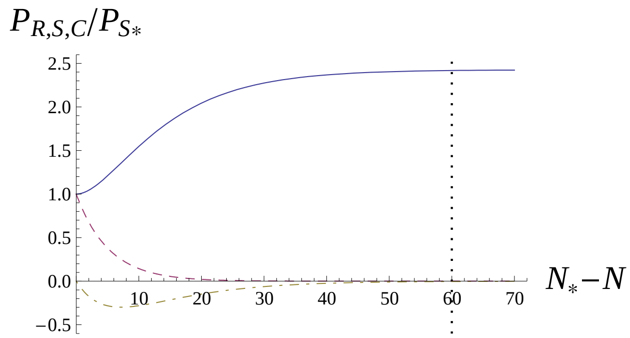

Figure 1: **The evolutions of power spectra with respect to e-folding number $N_* - N$ after crossing the horizon. The curvature power spectrum $\mathcal{P}_\mathcal{R}$ is signified by a blue solid line. The entropy power spectrum $\mathcal{P}_\mathcal{S}$ is denoted by a purple dashed line. The cross-correlation power spectrum $\mathcal{P}_\mathcal{C}$ is depicted by a brown dot-dashed line. All of the power spectra are normalized by $\mathcal{P}_{\mathcal{S}*}$, the entropy power spectrum at horizon-crossing. The vertical black dotted line corresponds to $N_* - N = 60$.**

uncertainties if $M_p^4/(2gV) \sim \mathcal{O}(\epsilon_1)$. To get a robust conclusion, one should take all these effects into account carefully, resorting to the numerical method.

Another issue of our analysis is the special limit we have taken: $M_p^2/\varphi^2 \ll \lambda m^2\varphi^2/M_p^2 \ll 1$. Taking such a limit is two-fold. On the one hand, it enables us to get a nearly scale-invariant entropy perturbation. On the other hand, we find $3H\dot{\varphi} \simeq m^2\varphi(\lambda m^2\varphi^4 - M_p^4)/M_p^4 > 0$ in this limit, hence the scalar field was not rolling down but rolling up the potential $V(\varphi)$. Also the Hubble parameter is forced to grow ($\epsilon_1 > 0$). The same thing also happens in "phantom" inflation (single-field inflation with a kinetic term of the wrong sign), see *e.g.* [38]. In our model, as the inflaton φ grows, the slow-roll parameter ϵ_1 tends to order unity and the inflation is expected to cease then. It is still unclear how to terminate inflation and trigger reheating in this model. We leave it as an open problem for future investigation. A more interesting problem is to get a more realistic, slow-roll model.

According to the result in [24], to the leading order, the power spectrum of tensor type perturbation in $f(\varphi, R)$ gravity is

$$\mathcal{P}_T \simeq \frac{H^2}{2\pi^2 F} = \mathcal{P}_{\mathcal{S}*}(2\delta_1 - 4\epsilon_1). \tag{90}$$

It is conserved after Hubble-exit, with a spectral index

$$n_T \simeq \frac{2\dot{H}}{H^2} - \frac{\dot{F}}{HF} = 2\epsilon_1 - \delta_1. \tag{91}$$

For the specific model (68), in the limit $M_p^2/\varphi^2 \ll \lambda m^2\varphi^2/M_p^2 \ll 1$, they are

$$\frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{S}*}} \simeq 4\epsilon_{1*}, \qquad n_T \simeq -2\epsilon_{1*}. \tag{92}$$

In our case, the tensor spectrum is red tilted, which is different from "phantom" inflation. One would have noticed that the entropy-to-scalar ratio is small at the end of inflation,

$$r_T = \frac{\mathcal{P}_T}{\mathcal{P}_\mathcal{R}} \simeq \frac{3364\epsilon_{1*}^2}{841\epsilon_{1*} + 12} \simeq 0.02. \tag{93}$$

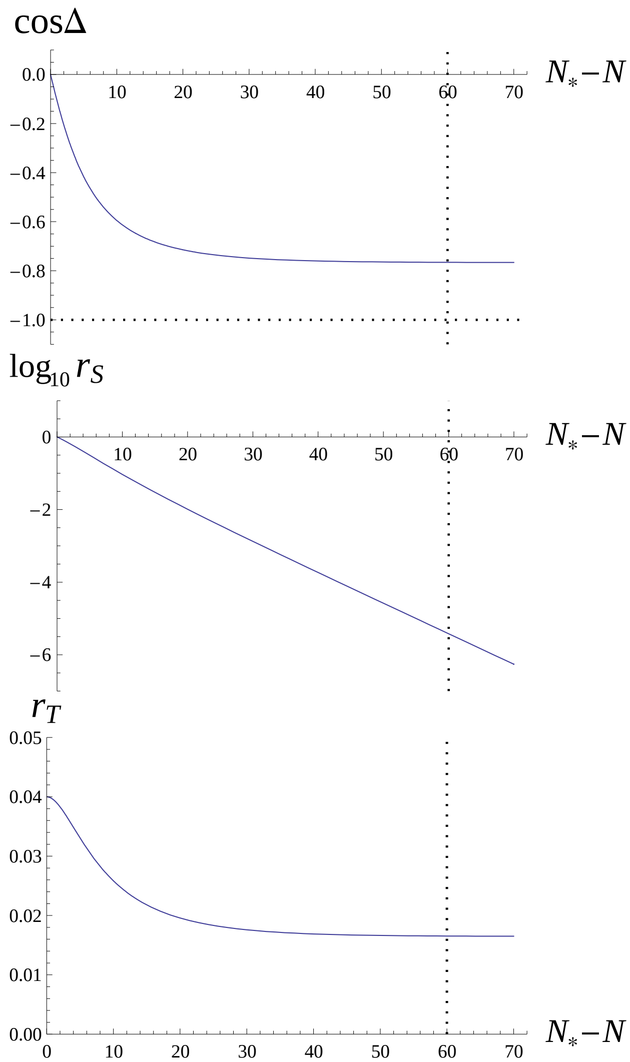

Figure 2: **The evolutions of correlation coefficient $\cos\Delta$ (the top graph), entropy-to-curvature ratio r_S (its logarithm, the middle graph) and tensor-to-scalar ratio r_T (the bottom graph) with respect to e-folding number $N_* - N$ after crossing the horizon. The vertical black dotted lines correspond to $N_* - N = 60$. The horizontal black dotted line corresponds to $\cos\Delta = -1$, that is, the totally anti-correlated situation.**

Its evolution curve outside the horizon is plotted in the lower graph of figure 2.

VI. SUMMARY

We investigated the cosmological perturbations in generalized gravity, where the Ricci scalar and a scalar field are coupled non-minimally by an arbitrary function $f(\varphi, R)$, with the Einstein gravity as a special limit. This general form unifies the usual modified gravity [39] and scalar-tensor theory, but often introduces an additional degree of freedom. In the FLRW background, by studying the first order perturbation theory, we decomposed the scalar type perturbations into the curvature perturbation and the entropy perturbation,

whose evolution equations are obtained. The effects of entropy perturbation in this class of model were seldom studied in the past.

Then we applied this framework to inflation theory. The slow-roll conditions and the quantized initial condition are discussed. The quantization of second order action will appear in a future work, which will put our discussion at the end of section IV on a firmer ground.

Analytically we studied two special examples: the $\delta s = 0$ models in subsection V A and the $g(\varphi)R^2$ model in subsection V B. In the former example we unified the previous known results in a unique form and extended them to the general case. The latter example is more interesting. In the model $g(\varphi) = \frac{1}{4}\lambda\varphi^2$, $V(\varphi) = \frac{1}{2}m^2\varphi^2$, we paid attention on the limit $M_p^2/\varphi^2 \ll \lambda m^2\varphi^2/M_p^2 \ll 1$. This limit helps us to get a nearly scale-invariant entropy perturbation. Although the entropy perturbation was large at the Hubble-exit, it decreased significantly outside the horizon. At the end of inflation, the "entropy-to-curvature ratio" we defined by (78) was of order 10^{-6}, well below the constraint of five-year WMAP data.

In spite of the above results, there are still some important problems unsolved. First, the quantization should be performed to confirm the normalization in (53) directly. Second, to get robust conclusions and richer phenomena, maybe one has to employ a numerical method to solve evolution equations (44) and (46) more generally. Third, the formalism we developed in section III can be applied to other cosmological stages and scenarios, such as the preheating stage and the curvaton scenario, *etc.* In [40, 41, 42, 43], different schemes of modified gravity were investigated mainly concerning their effects on the late time evolution of our universe, the possible effect of $f(\varphi, R)$ gravity on late universe is still waiting for us to study.

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Appendix A: TWO-FIELD MODEL AND CONFORMAL TRANSFORMATION

For comparison, let us collect here the well-known results for two-field inflation with a non-standard kinetic term, which is conformally equivalent to the $f(\varphi, R)$ generalized gravity [17, 18, 19, 24]. Usually the $f(\varphi, R)$ formulism is called Jordan frame while the non-standard two filed formulism is called Einstein fame. In this appendix, we use the super/subscripts "J" and "E" to distinguish different frames. The action two-field inflation with a non-standard kinetic term in Einstein fame is

$$S = \int d^4x\sqrt{-g_E}\left[\frac{1}{16\pi G}R_E - \frac{1}{2}g_E^{\alpha\beta}\partial_\alpha^E\chi\partial_\beta^E\chi - \frac{1}{2}e^{2b(\chi)}g_E^{\alpha\beta}\partial_\alpha^E\varphi\partial_\beta^E\varphi - V(\varphi,\chi)\right]. \quad \text{(A1)}$$

The reference [31] has an original and clear discussion on the curvature and entropy perturbations in this model. The readers may get details there. In a further study [13], it was clarified that the total entropy perturbation includes not only the relative entropy perturbation, but also an entropy perturbation proportional to k^2/a^2H^2. Although it will give a correction at the scale $k \sim aH$, its contribution to entropy perturbation is suppressed in long wavelength, so we do not consider its effects in our investigation.

In longitudinal gauge, the canonical variable in this frame is given by

$$v_\chi = a_E \left(\delta\chi + \frac{\partial_t^E \chi}{H_E} \psi_E \right), \qquad v_\varphi = a_E e^b \left(\delta\varphi + \frac{\partial_t^E \varphi}{H_E} \psi_E \right). \tag{A2}$$

They are related to the canonical variables for curvature and entropy perturbations

$$\begin{aligned} v_{\mathcal{R}} &= \frac{\partial_t^E \chi}{\sqrt{(\partial_t^E \chi)^2 + e^{2b}(\partial_t^E \varphi)^2}} v_\chi + \frac{e^b \partial_t^E \varphi}{\sqrt{(\partial_t^E \chi)^2 + e^{2b}(\partial_t^E \varphi)^2}} v_\varphi \\ &= \frac{a_E \sqrt{(\partial_t^E \chi)^2 + e^{2b}(\partial_t^E \varphi)^2}}{H_E} \mathcal{R}, \\ v_{\mathcal{S}} &= \frac{\partial_t^E \chi}{\sqrt{(\partial_t^E \chi)^2 + e^{2b}(\partial_t^E \varphi)^2}} v_\varphi - \frac{e^b \partial_t^E \varphi}{\sqrt{(\partial_t^E \chi)^2 + e^{2b}(\partial_t^E \varphi)^2}} v_\chi \\ &= \frac{a_E \sqrt{(\partial_t^E \chi)^2 + e^{2b}(\partial_t^E \varphi)^2}}{H_E} \mathcal{S}. \end{aligned} \tag{A3}$$

Making use of the conformal transformation

$$g_{\mu\nu}^E = \frac{F}{M_p^2} g_{\mu\nu}^J \tag{A4}$$

and the identification

$$\frac{\chi}{M_p} = \sqrt{\frac{3}{2}} \ln \frac{F}{M_p^2}, \tag{A5}$$

one can prove the following relations [18, 21, 24]

$$\begin{aligned} & a_E = \frac{\sqrt{F}}{M_p} a_J, \qquad dt_E = \frac{\sqrt{F}}{M_p} dt_J, \qquad H_E = \frac{M_p(2H_J F + \partial_t^J F)}{2F^{\frac{3}{2}}}, \\ & \psi_E = \frac{1}{2}(\phi_J + \psi_J), \qquad \delta\chi = \sqrt{\frac{3}{2}} M_p (\psi_J - \phi_J). \end{aligned} \tag{A6}$$

If we take

$$b(\chi) = -\chi/(\sqrt{6} M_p), \qquad V(\phi, \chi) = \frac{M_p^4}{2F^2}[R_J F(\varphi, R_J) - f(\varphi, R_J)], \tag{A7}$$

then action (1) can be perfectly reproduced from action (A1). For details of derivation, see reference [18]. In these frames, the conformal time are coincident $d\tau_E = d\tau_J$, so the canonical quantized variables (A3) are exactly equivalent to (52) by the above conformal transformation. This conformal equivalence is powerful. Given a specific function $f(\varphi, R_J)$, one can know the detailed form of $V(\varphi, \chi)$ using $F = \frac{\partial}{\partial R_J} f(\varphi, R_J)$ and (A5), and then do our job in the more familiar Einstein frame.[7]

[7] Thank the referee for an emphasis on this point.

Appendix B: A TRADITIONAL DEFINITION OF CURVATURE AND ENTROPY PERTURBATIONS

In all of our discussion, we take the curvature perturbation and entropy perturbation as defined in (18) and (20). But such a definition is different from the traditional one. If we regard the $f(\varphi, R)$ theory as the Einstein gravity with exotic matter contents induced by the non-minimal coupling, and following the spirit of [35], then we will arrive at a traditional form of curvature and entropy perturbations. Let us elaborate a little on this point. One should keep in mind that this point of view is different from that in appendix A. Although we also use the term "Einstein gravity" here, it does not mean the Einstein frame.

Formally, Friedmann equations (7) and (8) can be rewritten as

$$H^2 = \frac{1}{3M_p^2}\rho, \quad \dot{H} = -\frac{1}{2M_p^2}(\rho + p), \tag{B1}$$

with

$$\begin{aligned} \rho &= \frac{M_p^2}{F}(\frac{1}{2}\dot{\varphi}^2 + V + \frac{RF - f}{2} - 3H\dot{F}), \\ p &= \frac{M_p^2}{F}(\frac{1}{2}\dot{\varphi}^2 - V - \frac{RF - f}{2} + \ddot{F} + 2H\dot{F}). \end{aligned} \tag{B2}$$

Here in the effective energy density and pressure we have reckoned the contribution of the non-minimal coupling. Our final result seriously depends on this trick. The perturbations obey equations (10-13). Then the comoving curvature perturbation is given by

$$\mathcal{R}_{eff} = \psi - \frac{H}{\rho + p}\delta q = \psi - \frac{H}{\dot{H}}\left(\dot{\psi} + H\phi\right). \tag{B3}$$

The curvature perturbation on the uniform density hyper-surface is well defined,

$$\zeta_{eff} = -\psi - \frac{H}{\dot{\rho}}\delta\rho = -\psi + \frac{H}{\dot{H}}\left(\dot{\psi} + H\phi\right) - \frac{1}{3\dot{H}}\frac{\nabla^2}{a^2}\psi. \tag{B4}$$

At the same time, the entropy perturbation δs_{eff} is defined by

$$\begin{aligned} T\delta s_{eff} &= \delta p - c_s^2\delta\rho, \\ c_s^2 &= \frac{\partial p}{\partial\rho} = \frac{\dot{p}}{\dot{\rho}} = -\frac{3H\dot{H} + \ddot{H}}{3H\dot{H}}. \end{aligned} \tag{B5}$$

It would be interesting to notice that

$$\zeta_{eff} + \mathcal{R}_{eff} = \frac{2M_p^2}{3(\rho + p)}\frac{\nabla^2}{a^2}\psi. \tag{B6}$$

Finally, one can quickly prove

$$-\dot{\mathcal{R}}_{eff} = \frac{H}{\dot{H}}\left[\ddot{\psi} + H\dot{\phi} - \frac{\ddot{H}}{\dot{H}}(\dot{\psi} + H\phi) + 2\dot{H}\phi\right], \tag{B7}$$

and

$$-\frac{\dot{H}}{H}\dot{\mathcal{R}}_{eff} = \frac{1}{2M_p^2}(\delta p - c_s^2\delta\rho) + \frac{c_s^2}{a^2}\nabla^2\psi + \frac{1}{3a^2}\nabla^2(\psi - \phi)$$
$$= \frac{1}{2M_p^2}T\delta s_{eff} + \frac{c_s^2}{a^2}\nabla^2\psi + \frac{1}{3a^2}\nabla^2(\psi - \phi). \tag{B8}$$

Now we see that the curvature perturbation $\mathcal{R}_{eff}$ and entropy perturbation δs_{eff} are related in the traditional manner. However, the full expression of δs_{eff} is rather complicated, accordingly its evolution equation is even more difficult to get. Therefore, the perturbations presented in this appendix are not convenient in studying the generalized gravity. Although the definition here appears very natural if one takes $f(\varphi, R)$ gravity as an effective Einstein theory, a more convenient choice should be (18) and (20).

Appendix C: EVOLUTION OF ENTROPY PERTURBATION

From (18) and (20), we get

$$F(\dot{\phi}+\dot{\psi}) = -\frac{1}{2}(2HF+\dot{F})(\phi+\psi) + \frac{2F\dot{\varphi}^2+3\dot{F}^2}{2HF+\dot{F}}\left[\mathcal{R} - \frac{1}{2}(\phi+\psi)\right]. \tag{C1}$$

$$\phi - \psi = \frac{2\dot{F}}{2F\dot{\varphi}^2+3\dot{F}^2}\delta s + \frac{2\dot{F}}{2HF+\dot{F}}\left[\frac{1}{2}(\phi+\psi) - \mathcal{R}\right]. \tag{C2}$$

Now equations (15) and (16) can be rewritten in the form

$$\begin{aligned} F(\ddot{\phi}+\ddot{\psi}) &= \left(\frac{2F\ddot{\varphi}}{\dot{\varphi}} - HF - 3\dot{F}\right)(\dot{\phi}+\dot{\psi}) \\ &+ \left[\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F}) - (2HF+\dot{F})^\bullet\right](\phi+\psi) \\ &+ \left(\frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}} - \dot{\varphi}^2 - 3\ddot{F}\right)(\phi-\psi) + \frac{F}{a^2}\nabla^2(\phi+\psi), \end{aligned} \tag{C3}$$

$$\begin{aligned} F(\ddot{\phi}-\ddot{\psi}) &= \left(4HF - 3\dot{F} + \frac{2F\ddot{\varphi}}{\dot{\varphi}} - \frac{F^2F_{,\varphi}}{3F_{,R}\dot{\varphi}}\right)(\dot{\phi}+\dot{\psi}) - 3HF(\dot{\phi}-\dot{\psi}) \\ &+ \left[2F(2H^2+\dot{H}) - (2HF+\dot{F})^\bullet + (2HF+\dot{F})\left(\frac{\ddot{\varphi}}{\dot{\varphi}} - \frac{FF_{,\varphi}}{6F_{,R}\dot{\varphi}}\right)\right](\phi+\psi) \\ &+ \left[2F(2H^2+\dot{H}) + \frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}} - \dot{\varphi}^2 - 3\ddot{F} - \frac{F^2}{3F_{,R}} - \frac{F\dot{F}F_{,\varphi}}{2F_{,R}\dot{\varphi}}\right](\phi-\psi) \\ &+ \frac{2F}{3a^2}\nabla^2(\phi+\psi) + \frac{F}{a^2}(\phi-\psi). \end{aligned} \tag{C4}$$

Differentiating (20) with respect to time and making use of (C3), one obtains

$$
\begin{aligned}
\dot{\delta s} \;=\; & \left(\frac{2F\ddot{\varphi}}{\dot{\varphi}} - \frac{3}{2}\dot{F}\right)(\dot{\phi}+\dot{\psi}) + \frac{2F\dot{\varphi}^2+3\dot{F}^2}{2\dot{F}}(\dot{\phi}-\dot{\psi}) \\
& + \left[\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F}) - \frac{1}{2}(2HF+\dot{F})^{\bullet}\right](\phi+\psi) \\
& + \left[\left(\frac{F\dot{\varphi}^2}{\dot{F}}\right)^{\bullet} + \frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}} - \dot{\varphi}^2 - \frac{3}{2}\ddot{F}\right](\phi-\psi) \\
& + \frac{F}{a^2}\nabla^2(\phi+\psi),
\end{aligned}
\tag{C5}
$$

which gives

$$
\begin{aligned}
& \frac{2F\dot{\varphi}^2+3\dot{F}^2}{2\dot{F}}(\dot{\phi}-\dot{\psi}) \\
=\; & \dot{\delta s} + \left(\frac{3}{2}\dot{F} - \frac{2F\ddot{\varphi}}{\dot{\varphi}}\right)(\dot{\phi}+\dot{\psi}) + \left[\frac{1}{2}(2HF+\dot{F})^{\bullet} - \frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F})\right](\phi+\psi) \\
& + \left[\dot{\varphi}^2 + \frac{3}{2}\ddot{F} - \left(\frac{F\dot{\varphi}^2}{\dot{F}}\right)^{\bullet} - \frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}}\right](\phi-\psi) - \frac{F}{a^2}\nabla^2(\phi+\psi).
\end{aligned}
\tag{C6}
$$

Taking the derivative of (C5) once more, we have

$$
\begin{aligned}
\ddot{\delta s} \;=\; & \left(\frac{2F\ddot{\varphi}}{\dot{\varphi}} - \frac{3}{2}\dot{F}\right)(\ddot{\phi}+\ddot{\psi}) + \frac{2F\dot{\varphi}^2+3\dot{F}^2}{2\dot{F}}(\ddot{\phi}-\ddot{\psi}) \\
& + \left[\left(\frac{2F\ddot{\varphi}}{\dot{\varphi}}\right)^{\bullet} + \frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F}) - (HF+2\dot{F})^{\bullet}\right](\dot{\phi}+\dot{\psi}) \\
& + \left[\left(\frac{2F\dot{\varphi}^2}{\dot{F}}\right)^{\bullet} + \frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}} - \dot{\varphi}^2\right](\dot{\phi}-\dot{\psi}) \\
& + \left\{\left[\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F})\right]^{\bullet} - \frac{1}{2}(2HF+\dot{F})^{\bullet\bullet}\right\}(\phi+\psi) \\
& + \left[\left(\frac{F\dot{\varphi}^2}{\dot{F}}\right)^{\bullet\bullet} + \left(\frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}} - \dot{\varphi}^2 - \frac{3}{2}\ddot{F}\right)^{\bullet}\right](\phi-\psi) \\
& + \frac{F}{a^2}\nabla^2(\dot{\phi}+\dot{\psi}) + (\dot{F}-2HF)\frac{\nabla^2}{a^2}(\phi+\psi).
\end{aligned}
\tag{C7}
$$

Substituting (C3) and (C4), (C6), (C1) and (C2) into (C7) step by step to eliminate $\ddot{\phi}+\ddot{\psi}$ and $\ddot{\phi}-\ddot{\psi}$, $\dot{\phi}-\dot{\psi}$, $\dot{\phi}+\dot{\psi}$ and $\phi-\psi$ respectively, one will arrive at a rather scattering equation:

$$
\begin{aligned}
\ddot{\delta s} \;=\; & \left\{\left[\ln\frac{\dot{\varphi}^2(2F\dot{\varphi}^2+3\dot{F}^2)}{\dot{F}^2}\right]^{\bullet}-3H\right\}\dot{\delta s}+\frac{\nabla^2}{a^2}\delta s \\
& +\left\{\left[\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F})\right]^{\bullet}-\frac{1}{2}(2HF+\dot{F})^{\bullet\bullet}+\left(\frac{2\ddot{\varphi}}{\dot{\varphi}}-\frac{3\dot{F}}{2F}\right)\left[\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F})-(2HF+\dot{F})^{\bullet}\right]\right. \\
& +\left(\frac{\dot{\varphi}^2}{\dot{F}}+\frac{3\dot{F}}{2F}\right)\left[2F(2H^2+\dot{H})-(2HF+\dot{F})^{\bullet}+(2HF+\dot{F})\left(\frac{\ddot{\varphi}}{\dot{\varphi}}-\frac{FF_{,\varphi}}{6F_{,R}\dot{\varphi}}\right)\right] \\
& \left.+\left[\left(\ln\frac{\dot{\varphi}^2(2F\dot{\varphi}^2+3\dot{F}^2)}{\dot{F}^2}\right)^{\bullet}-3H\right]\left[\frac{1}{2}(2HF+\dot{F})^{\bullet}-\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F})\right]\right\}(\phi+\psi) \\
& +\left\{\left(\frac{2F\ddot{\varphi}}{\dot{\varphi}}\right)^{\bullet}+\frac{\ddot{\varphi}}{\dot{\varphi}}(2HF+\dot{F})-(HF+2\dot{F})^{\bullet}+\left(\frac{2\ddot{\varphi}}{\dot{\varphi}}-\frac{3\dot{F}}{2F}\right)\left(\frac{2F\ddot{\varphi}}{\dot{\varphi}}-HF-3\dot{F}\right)\right. \\
& +\left(\frac{\dot{\varphi}^2}{\dot{F}}+\frac{3\dot{F}}{2F}\right)\left(4HF-3\dot{F}+\frac{2F\ddot{\varphi}}{\dot{\varphi}}-\frac{F^2F_{,\varphi}}{3F_{,R}\dot{\varphi}}\right) \\
& \left.+\left[\left(\ln\frac{\dot{\varphi}^2(2F\dot{\varphi}^2+3\dot{F}^2)}{\dot{F}^2}\right)^{\bullet}-3H\right]\left(\frac{3}{2}\dot{F}-\frac{2F\ddot{\varphi}}{\dot{\varphi}}\right)\right\} \\
& \times\left\{-\frac{2HF+\dot{F}}{2F}(\phi+\psi)-\left(\frac{\dot{\varphi}^2}{\dot{F}}+\frac{3\dot{F}}{2F}\right)\frac{2\dot{F}}{2HF+\dot{F}}\left[\frac{1}{2}(\phi+\psi)-\mathcal{R}\right]\right\} \\
& +\left\{\left(\frac{F\dot{\varphi}^2}{\dot{F}}\right)^{\bullet\bullet}+\left(\frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}}-\dot{\varphi}^2-\frac{3}{2}\ddot{F}\right)^{\bullet}+\left(\frac{2\ddot{\varphi}}{\dot{\varphi}}-\frac{3\dot{F}}{2F}\right)\left(\frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}}-\dot{\varphi}^2-3\ddot{F}\right)\right. \\
& +\left(\frac{\dot{\varphi}^2}{\dot{F}}+\frac{3\dot{F}}{2F}\right)\left[2F(2H^2+\dot{H})+\frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}}-\dot{\varphi}^2-3\ddot{F}-\frac{F^2}{3F_{,R}}-\frac{F\dot{F}F_{,\varphi}}{2F_{,R}\dot{\varphi}}\right] \\
& \left.+\left[\left(\ln\frac{\dot{\varphi}^2(2F\dot{\varphi}^2+3\dot{F}^2)}{\dot{F}^2}\right)^{\bullet}-3H\right]\left[\dot{\varphi}^2+\frac{3}{2}\ddot{F}-\left(\frac{F\dot{\varphi}^2}{\dot{F}}\right)^{\bullet}-\frac{3\dot{F}\ddot{\varphi}}{\dot{\varphi}}\right]\right\} \\
& \times\left\{\frac{2\dot{F}}{2F\dot{\varphi}^2+3\dot{F}^2}\delta s+\frac{2\dot{F}}{2HF+\dot{F}}\left[\frac{1}{2}(\phi+\psi)-\mathcal{R}\right]\right\} \\
& +\left\{-\frac{1}{2}(2HF+\dot{F})+\dot{F}-2HF+\frac{2F\ddot{\varphi}}{\dot{\varphi}}-\frac{3}{2}\dot{F}+\frac{2F\dot{\varphi}^2}{3\dot{F}}+\dot{F}\right. \\
& \left.-F\left[\ln\frac{\dot{\varphi}^2(2F\dot{\varphi}^2+3\dot{F}^2)}{\dot{F}^2}\right]^{\bullet}+3HF\right\}\frac{\nabla^2}{a^2}(\phi+\psi).
\end{aligned}
\tag{C8}
$$

This equation looks terribly lengthy. However, repeatedly employing the relation (8) or namely

$$(2HF+\dot{F})^{\bullet}=3H\dot{F}-\dot{\varphi}^2, \tag{C9}$$

after careful calculation, we find the coefficients of the $\frac{2\dot{F}}{2HF+\dot{F}}\left[\frac{1}{2}(\phi+\psi)-\mathcal{R}\right]$ and remaining $(\phi+\psi)$ terms are exactly vanished, resulting in a much simpler form (27).

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